1 Lots of stars

A thermal emitter of temperature T obeys the law:

$$L = A\sigma T^4 \tag{1}$$

where L is the star's luminosity, σ is a constant of proportionality and A is the surface area of the emitter. Stars are spherical thermal emitters. The area of a sphere is $A = 4\pi R^2$, which we can combine with the first equation to give $L_{star} = 4\pi R^2 \sigma T^4$.
Furthermore, the emitter's spectrum will *peak* at a wavelength λ_{max} given by:

$$\lambda_{max} = \frac{.002898m}{T}. \tag{2}$$

Hint: remember to look at ratios; you can do this by dividing.

1. Star 1 is *hotter* and *brighter* than star 2. Can we say which star is larger?
   No, we cannot determine which star is larger. The luminosity depends on the radius *and* the temperature. We know star 1 is brighter and hotter. *Maybe* star 1 is also larger, in which case both its radius and temperature contribute to making it brighter than star 2. *Or*, maybe star 1 is smaller than star 2, but its so hot that it makes up for being smaller by a large increase in its temperature. So, in this case you can't be sure which star is larger until I give you specific values.

2. At their respective peak wavelengths, which star (1 or 2) has shorter wavelength photons? Which star has higher frequency photons (again, at their respective peak wavelengths)?

   The peak wavelength is given by Eq. 2. From this equation, we can see that as you *increase* the temperature of the star, the peak wavelength *decreases* (if this isn't clear, try plugging in a random number for T, then plug in a *bigger* random number; you should see that λ_{max} gets smaller). Since Star 1 is hotter, it has a larger T and a smaller λ_{max}. From the wave equation of light, $c = \lambda f$, we see that the larger the wavelength (λ), the smaller the frequency. Since Star 1 has a smaller λ_{max}, it must then have the *larger* frequency at the peak wavelength. Always remember – the larger the wavelength, the smaller the frequency and vice-versa.

3. Which star has higher energy photons at their peak wavelength?

   From the equation $E = hf$, we can see that the energy of a photon is directly proportional to the frequency of the photon. Since Star 1 has higher frequency photons at its peak, it will also have higher energy photons.

4. $*$ Star 3 is twice as hot as star 4 but three times fainter. What is the ratio of the radii of the two stars?

   From Eq. 1 we have $L = 4\pi R^2 \sigma T^4$. So Star 3 has $L_3 = 4\pi R_3^2 \sigma T_3^4$. Similarly, Star 4 has $L_4 = 4\pi R_4^2 \sigma T_4^4$. Now, let's divide these two equations:

   $$\frac{L_3}{L_4} = \frac{4\pi R_3^2 \sigma T_3^4}{4\pi R_4^2 \sigma T_4^4} \tag{3}$$

   There's a factor of 4π and σ on the top and bottom, so we can cancel these. The equation simplifies to:

   $$\frac{L_3}{L_4} = \frac{R_3^2 T_3^4}{R_4^2 T_4^4} \tag{4}$$

   Now we have a neat expression relating the brightnesses, radii, and temperature! We don't have actual values for any of these variables, but we do have their values *in relation to on another*! I gave you that star 3 is twice as hot as star 4. Mathematically, this is simply $T_3 = 2T_4$. Likewise, I gave you that star 3 is three times fainter than star 4. Again, this is simply $L_3 = \frac{1}{3}L_4$. Let's plug these two expressions in:

   $$\frac{\frac{1}{3}L_4}{L_4} = \frac{R_3^2 (2T_4)^4}{R_4^2 T_4^4} = \frac{R_3^2}{R_4^2}\frac{16T_4^4}{T_4^4} \tag{5}$$

We see that L_4 cancels on the left, and T_4^4 cancels on the right. So, simplifying:

$$\frac{1}{3} = \frac{16R_3^2}{R_4^2} \tag{6}$$

Remember, we're looking for the ratio, $\frac{R_3}{R_4}$. So, we divide by 16 on both sides to isolate the terms involving the radius, and then take a square root:

$$\frac{1}{3\times 16} = \frac{R_3}{R_4} = \frac{1}{\sqrt{48}} \tag{7}$$

You can leave your answer like that, or compute $1/\sqrt{48} \approx 1/6.9$. So, star 3 has a radius about one-seventh that of star 4. That was a lot of math! This is a pretty tricky ratio problem, but if you can solve this problem you should be able to handle almost any problem involving ratios.

2 Let there be light!

When a photon – a little "packet" of light – hits an atom, it can bump one of the atom's electrons up in energy. To bump a photon up between two energy levels, the photon must have *exactly* the energy difference between the two levels. If it has too much or too little, the electron will stay put. Alternatively, if the photon has enough energy to *completely* remove the electron (known as "ionization") from the atom.

1. In the hydrogen atom, the difference in energy between successive energy levels gets smaller and smaller - that is, $E_2 - E_1 > E_3 - E_2 > E_4 - E_3$... and so on. It's very useful to visualize this: briefly sketch this setup by drawing a horizontal line for each energy level, with higher energy levels higher up on the page.

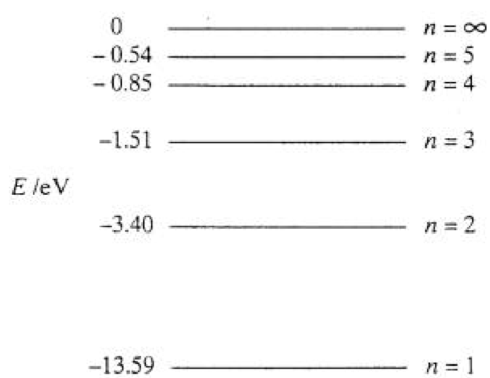

Figure 1: **Representation of hydrogen's energy levels. The amount of vertical space between successive lines indicates how much of an energy difference there is between the two levels. As you can see, the lower energy levels are spaced far apart (large energy gap between them), while the higher energy levels have close to the same energy. You don't need any of the numbers on here!**

2. For hydrogen, the energy levels, $E_n \propto -\frac{1}{n^2}$, where n is the number of the energy level. What is the ratio of the energy of an electron in the 6th energy level to one in the 2nd energy level?

   Remember, whenever you see a $\propto$ symbol, just rewrite the equation with an equality (= symbol), and multiply by a constant of proportionality (I'll call it k). In this case, $E_n = k \times \frac{1}{n^2}$. Then, for the 6th energy level we have $E_6 = k\frac{1}{6^2}$ and for the 2nd level $E_2 = k\frac{1}{2^2}$. We care about the *ratio* of E_6 and E_2, which is just a fancy way of saying we want E_6/E_2:

$$\frac{E_6}{E_2} = \frac{k\frac{1}{6^2}}{k\frac{1}{2^2}} = \frac{\frac{1}{36}}{\frac{1}{4}} \tag{8}$$

To simplify, multiply by 4 on the top and bottom to cancel out the $\frac{1}{4}$ in the denominator (we can do this because multiplying by 4 on the top and bottom is equivalent to multiplying by one). We end up with:

$$\frac{E_6}{E_2} = \frac{1}{9} \tag{9}$$

3 Cop chase

Not many people came to my office hours so I wandered and pulled off an armored car heist. Unfortunately, the guards called the cops who started driving in my direction. During this high-speed chase, I was plagued by a few questions, listed below. One equation you may find useful:

$$\frac{f_{\text{receiver}} - f_0}{f_0} = \frac{v_{\text{source}} - v_{\text{receiver}}}{v_{\text{wave}}} \tag{10}$$

1. As I drove off with my money, the cops close on my tail and bridging the gap between us, their sirens began to sound different. Why did I hear such a change? Give an *intuitive* explanation. Try not to use any math; draw a picture if you can.

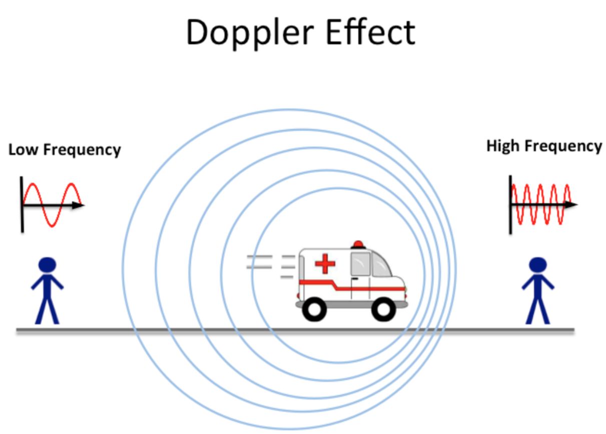

Figure 2: **The cop car (or in this picture, the ambulance) is sending out (sound) waves in my direction (I'm the observer on the right). But, since the car is moving *towards* me, these waves get pressed together. So, I'll receive one part of the wave after another *faster* than normal, which means I'll perceive a *higher* frequency. Alternatively, you can look at the space between the circles – for the observer on the right, the circles are pressed together and the wavelength is shorter. So, as the cops chase after me, I will hear a *higher* pitch (pitch is equivalent to the frequency of the sound wave) siren.**

2. I was driving at 47 m/s, and the cops at 50 m/s. How much did the frequency of the siren change? The speed of sound is 300m/s and the frequency of a police siren is 750 Hz.
   Here, we can simply plug into the formula I gave you. $v_{\text{source}} = 50m/s$ is the speed of the cop car, which is emitting the sound. $v_{\text{receiver}} = 47m/s$ is my speed. $v_{\text{wave}} = 300m/s$ is the speed of the sound wave. f_0 is the "rest," or original frequency of the siren. Solving for $f_{\text{receiver}} - f_0$ by multiplying both sides by f_0:

$$f_{\text{receiver}} - f_0 = f_0 \times \frac{v_{\text{source}} - v_{\text{receiver}}}{v_{\text{wave}}} = 750Hz \times \frac{47 - 50m/s}{300m/s} = 7.5Hz \tag{11}$$

   So the frequency *increases* – the math agrees with our intuitive explanation from part 1!

3. Is this case analogous to a "redshift" or a "blueshift"?
   When dealing with light (which is also subject to the doppler effect, since it is a wave), we call it a "blueshift" when the light wave is shifted to *smaller* wavelengths due to the doppler effect (blue light has a shorter

wavelength than red light). This occurs when the source is moving *towards* the receiver (as we have here!). Likewise, in this problem we found the police siren sound wave to be doppler-shifted to higher frequencies. So, this case is analogous to a blueshift. Of course, it's not an actual blueshift since we're dealing with sound, not light.

4 A smattering of bonus questions

These questions are best discussed in person, so I won't put solutions here. We may discuss one or two of these in section – for solutions to the rest, talk to me (in section or office hours) and I'm happy to explain them.

These are listed in order of *decreasing* relevance.

1. An electron in the E_5 state of hydrogen drops down one energy level. In the process, it emits a photon. Later, this photon runs into another hydrogen atom, with an electron in some unknown energy state. Could the photon ionize the atom's electron? Could it excite the electron up to a higher energy states? If so, what transition(s) will occur?

2. Do you think I was lying about my police story? Why or why not? *hint:* think about the magnitude of the frequency change.

3. Consider a star moving in a circular orbit, such that at some times the star is moving directly away from us, and at others it is moving directly towards us. We use the doppler shift of the star's spectrum to infer the velocity at many different times. *Sketch* what the velocity-time graph looks like.