1 Astro 7A, Week 5

1. **Practice with optical depth**

   (a) Suppose you are sitting under the shade of a leafy tree. What is the optical depth of photons coming from above the tree to you? **It would be kind of a useless tree if it were optically thin with $\tau \ll 1$. Could it be optically thick with $\tau \gg 1$? Seems unlikely to me – most of the time when you are sitting in the shade you still get SOME light – it's not pitch black! So I would estimate $\tau \sim 1$. Remember $I = I_0 e^{-\tau}$, so when $\tau \sim 1$ about half ($1/e \sim 1/2.7 \sim 0.5$) the light is getting through.**

   (b) When you are staring into, say, a body of water, you can only see so far down. Roughly to what optical depth can you see? (hopefully this gives more motivation for thinking of optical depth as just another coordinate system!) **In general, the photons we are receiving emerge from the point where $\tau = 1$. That is, once τ drops down to 1, photons can (on average) travel all the way to the observer unimpeded – you show this in the question two parts down. This is a VERY important point to understand (for, like, all of astrophysics)!!**

   (c) In the absence of any extinction, a certain object has apparent magnitude m. Now suppose I magically sprinkle some dust in between you and the object, such that the line-of-sight optical depth is $\tau \sim 1$. What is the new apaprent magnitude m' in terms of m? This is a handy relation to keep in your back pocket. **amount of light you receive drops off as $e^{-\tau}$, so for $\tau \sim 1$ you receive about 60% of the light from before. then $\Delta m = -2.5 \log_{10} I_f/I_0$ with $I_f/I_0 = e^{-1} \approx 0.3$. So $\Delta m = -2.5 \log_{10}(0.3) \approx 1$ (actually 1.08). So an optical depth of unity corresponds to a drop in magnitude of roughly 1.**

   (d) Here is another way to think of the $\tau \sim 1$ condition. In class we derived the expression for the mean free path $\lambda_{mfp} = 1/n\sigma$. For an optical depth of unity, what is the path length s (assume n and σ are constant)? How does this comapre to λ_{mfp}? **$\tau = n\sigma s = s/\lambda \rightarrow s = \lambda$ at $\tau = 1$. Recall that s is the line-of-sight distance traveled, so this is saying that when τ reaches 1, your mean free path – the length you can travel before running into something – is equal to the line of sight distance you need to travel. In other words, a photon starting at $\tau = 1$ can get to you, the observer, without running into anything (on average). This is why the first photons we can see are those emerging from $\tau = 1$ (photons starting from $\tau \gg 1$ will certainly run into something on the way and get scattered).**

   (e) Suppose that you are observing a certain star 10 pc away. Just at the star, the number density of electrons is $n_0 = 10$ cm^{-3}. Suppose the number density of electrons falls off as x^{-2} (where x is the distance as measured from the star)[1]. Calculate the optical depth of photons from the star to you. The cross section for the interaction of photons and electrons is given by the Thompson cross section $\sigma_T \sim 10^{-24}$ cm^{-2}. **More generally, $\tau = \int n\sigma dx$ where in this case $n = n(x)$ and $\sigma = \sigma_T$ is a constant. So in this case we have $\tau \propto \sigma_T \int x^{-2} dx$. Integrate away.**

   (f) In class we derived an equation for how intensity falls off with optical depth $I = I_0 e^{-\tau}$. This expression comes from the differential equation $\frac{dI}{d\tau} = -I$. What this equation is implicitly saying is that, on the way to us, light is only absorbed (I can only get smaller). But in the real world, there can be stuff in the way. And you already know (from last week) that stuff can GLOW. So this ADDS photon to your line of sight! Let's account for this in the previous differential equation as follows:

   $$\frac{dI}{d\tau} = -I + S, \tag{1}$$

   where S is the intensity we pick up along the way (note that in general S is not a constant!). Derive a NEW expression for $I(\tau)$ in terms of the initial intensity ("backlight" intensity) I_0 and S. For simplicity, assume that S is spatially constant. Once you have your final expression, take the limit $\tau \ll 1$ and $\tau \gg 1$. Does your answer make sense?

[1] This is an entirely unrealistic problem.

I don't want to get bogged down in the math, so first let me quote the answer

$$I(\tau) = I_0 e^{-\tau} + S\left(1 - e^{-\tau}\right). \tag{2}$$

Now, the interesting part. What happens when $\tau \ll 1$? Then we have $e^{-\tau}$ with $\tau \ll 1$ which should give you a Pavlovian reaction to TAYLOR EXPAND:

$$e^{-\tau} \approx 1 - \tau \text{ for } \tau \ll 1. \tag{3}$$

Therefore, the first term becomes $I_0 e^{-\tau} \approx I_0(1-\tau) \approx I_0 - I_0\tau$. But since $\tau \ll 1$, we can just approximate $I_0 - I_0\tau \approx I_0$. What about the second term? We have $S(1 - e^{-\tau})$ – plugging in the same Taylor expansion the term reduces to $S(1-(1-\tau)) = S\tau$. Therefore $I(\tau) \approx I_0 + S\tau$ when $\tau \ll 1$. Does this make sense? Yes! If the medium is optically thin all light passes through, then we should definitely *at least* get the light that we started with I_0 back. But, now the medium is "glowing" – so we get an additional contribution $S\tau$ to the light that we see. Now again for $\tau \gg 1$. Then the exponential in the first term $e^{-\tau} \to 0$ and so it kills off the I_0 term. What about the second term involving S? Again, the exponential term here goes to zero, so $1 - e^{-\tau} \to 1$ and we are left with $I(\tau) \approx S$. This also makes sense. When the medium is optically thick, we don't receive any of the light that we started with (this is the I_0 term being killed off). But now the medium is glowing, so we at least get the additional contribution S.

2. **Probability density functions** Consider a probability density function $\psi(x) \propto x^2$.

   (a) Normalize the probability to unity over the range [0,10] (i.e., determine the constant k such that the integral of kx^2dx is unity over the desired range). **Normalization constraint: $1 = \int_0^{10} kx^2dx$. k, is a constant, just solve. Comment: with the MB distribution you may have seen one or two normalization constraints. One is in terms of the total number of particles:**

   $$n(v) = n_{\text{tot}} \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 \exp(-mv^2/kT), \tag{4}$$

   the other is in terms of a fraction/probability of particles:

   $$f(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 \exp(-mv^2/kT). \tag{5}$$

   Notice that they are EXACTLY the same, just in the latter I divided out by n_{tot}, which makes sense if you want to talk in terms of a fraction of particles with speed v. These are totally equivalent statements (with different integral constraints).

   (b) Suppose that $\psi(x)$ represents the probability density for scores on a test. If there are one thousand students in the class, how many do we expect to have a score between [0,1]? How about [9,10]? **The fraction of scores we expect over some range [a,b] is $f_{ab} = \int_a^b \psi dx = \int_a^b kx^2dx$**

   (c) What do we expect the average score to be? **Given a probability density function $\psi(x)$ (which tells you how likely it is to get a given value of x) and we want to know the average of some function $g(x)$, the average value is given by: $\int \psi(x)g(x)dx$. In this case, x is just a given score, so $g(x) = x$. Then the average score is just given by $\int_0^{10} \psi(x)xdx$.**

   (d) What about the average SQUARED score? Compare to the square of the average score. **Now we want to know the average of the SQUARED score, so the average of x^2. Then, we proceed in analogous fashion: the average squared score is equal to $\int_0^{10} \psi(x)x^2dx$.**

3. **Practice with Maxwell-Boltzmann**

   (a) *Draw* the MB distribution for two different temperatures $T_2 > T_1$. **Higher T curve is shifted to the right (higher velocities) and is BROADER. Make sure you convince yourself this is true by analyzing the mathematical expression for the MB distribution.**

(b) Determine the analogous version of the Maxwell-Boltzmann distribution, but for the total MOMENTUM of particles, i.e. instead of $f(v)$ determine $f(p)$. **We use the fact that $f(p)dp = f(v)dv$ – this should sound familiar, looks just like what we use to convert between F_λ and F_ν ($F_\lambda d\lambda = F_\nu d\nu$). It's just a conservation statement (in this case, conservation of number of particles instead of conservation of energy). Anyways, we now have that $f(p) = f(v)dv/dp$. Recall that $p = mv$ so that $dp/dv = m$. Therefore, we have $f(p) = f(v)/m$. For $f(v)$ we can j**

(c) Same as above, but for ENERGY. **As above, but now $f(E)dE = f(v)dv \rightarrow f(E) = f(v)dv/dE$. $E = 0.5mv^2$ and so $dE = mvdv$.**

(d) Suppose I wanted to determine an expression for the probability of finding a particle with speed less than some given speed v. Write down an expression with appropriate bounds (no need to actually do the integral, unless you really love doing integration) for this quantity. We call this the "cumulative" distribution function. $C(v) = \int_0^v f(v)dv$.