1 Astro 7A, Week 3

1. **Telescopes**

   (a) Consider the planned Thirty Meter Telescope (TMT) and your typical 10 inch at-home telescope. Suppose the TMT collects $N = 10^5$ photons after staring at an object for $\Delta t = 100$ seconds. How long would you have to stare with your at-home telescope to collect the same number of photons? **The light gathering power (LGP) $\propto D^2$. The number of photons you collect is proportional to the light gathering power (i.e., number of photons per second you collect) multiplied by the TIME that you're staring at the object. In other words:** $N_{\rm phot} \propto {\rm LGP}\Delta t$

   $$\frac{N_{\rm phot,2}}{N_{\rm phot,1}} = \frac{D_2^2}{D_1^2}\frac{\Delta t_2}{\Delta t_1}. \tag{1}$$

   We have $D_1 = 30$m and $\Delta t_1 = 100$ seconds. Also $D_2 = 10$ inches. Solve for Δt_2!

   (b) If I am observing with the TMT at 500 nm, what wavelength would I have to observe at with my at-home telescope to achieve a comparable diffraction limit (read: resolution)? Comment on whether this would even be possible with your at-home telescope.

   $\theta_c \sim \lambda/D$. **For the resolutions to be the same, we require the θ_c of both telescopes to be equal:**

   $$\frac{\lambda_1}{D_1} = \frac{\lambda_2}{D_2}. \tag{2}$$

   We have $\lambda_1 = 500$ nm and $D_1 = 30$ m and $D_2 = 10$ inches. Solve for λ_2.

   (c) Globular clusters are clusters of stars with $\sim 10^5$ stars confined to a region of ~ 1 pc. (Most) globular clusters in our galaxy live in the far reaches of the galaxy, at about $\sim$20 kpc from the Galactic center (our Solar System is located about 8 kpc from the center). Will the TMT be able to tell apart individual stars in a globular cluster in our galaxy? How about in a galaxy $d = 20$ Mpc away? **To tell if the TMT will be able to tell apart individual stars, we need to calculate the typical $\theta_{\rm sep}$ of stars in the cluster. $\theta_{\rm sep} \sim s/d$. What is the typical physical separation s? A good rule of thumb is that a typical interparticle spacing in a system is roughly $n^{-1/3}$, where n is the number density. We can calculate the number density as $n \sim N/R^3$. Therefore $n^{-1/3} \sim 0.02$ pc. So we have to calculate $\theta_{\rm sep} \sim 0.02{\rm pc}/20{\rm kpc}$. We have to compare this $\theta_{\rm sep}$ to θ_c for the TMT to determine if we will be able to resolve individual stars in the cluster. If $\theta_{sep} < \theta_c$ then the stars will be unresolved and they will all blur together. If vice-versa, you will be able to actually resolve individual stars in the cluster.**

   (d) Suppose I observe a galaxy of diameter $D = 1$ kpc at a distance away $d = 1$ Mpc. If the TMT will have a plate scale of 0.06 arcseconds/pixel, over how many pixels will the galaxy image be spread out?

   (e) Stars Alex and Ben are *exactly* the same, except that Star Ben has half the diameter of Star Alex. Suppose we use Telescope A to collect photons from Star Alex, and register 10 photons per second. How much larger must the diameter of Telescope B be to observe Star Ben and still register 10 photons per second? **The luminosity is $\propto R^2$. Since Ben has half the diameter it has one-fourth the luminosity. To make up for this, we need Telescope B to be larger. The light gathering power is $\propto D^2$, therefore we need a telescope that is twice as large.**

   (f) I put a filter on my telescope. Suppose the sensitivity function of the filter is Gaussian, centered on λ_1

   $$S_\lambda = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\lambda-\lambda_1)^2}{2\sigma^2}\right). \tag{3}$$

   If I shine light onto the telescope with a flat spectrum $F_\lambda = k$, write down an expression for the total flux through the filter.

   $$F_{\rm filter} = \int_{\lambda_1}^{\lambda_2} S_\lambda k d\lambda. \tag{4}$$

Just do the integral with S_λ as given.

(g) Does the colour of the object depend on how far away I put it[1]? **No – light falls off as $1/r^2$ independent of the wavelength of the photon. Therefore even though the object appears to dim as $1/r^2$, all wavelengths suffer equally, and therefore the relative amounts of flux in each waveband (i.e., the colour) stay the same.**

2. **General Kepler**

(a) Consider two planets with periods P_1 and P_2 (same central mass $M_\star$). What is the ratio of their semi-major axes? $P \propto a^{3/2}$ **by Kepler III therefore** $P_1/P_2 = (a_1/a_2)^{3/2}$**.**

(b) Consider two planets of mass m_1 and m_2 both with the same a around the same object of mass $M_\star \gg m_1, m_2$. What is the ratio of their periods? $P \propto 1/(M_\star + m_P)$**. But** $M_\star \gg m_P$ **so the periods are the same, to first order. But if you look in detail and account for the mass of the planet, then the more massive planet will have a slightly *shorter* period.**

(c) Derive an expression for the angular frequency ω of an object of mass m in circular orbit around $M \gg m$. And again for an elliptical orbit. $\omega = \sqrt{GM/r^3}$**. Useful expression to keep in your back pocket.**

(d) Halley's comet is a regular comet with eccentricty $e = 0.967$ and its closest approach to the Sun is 0.59 AU. What is Haley's comet's *greatest* distance from the Sun? Semi-major axis? **The cloest approach is periapse and is given by:** $r_p = (1-e)a \rightarrow a = r_p/(1-e)$ **which allows us to solve for** a**. Then plug into equation for apoapse, the farthest distance:** $r_a = (1+e)a = r_p(1+e)/(1-e)$**.**

(e) Show that the total energy of a Keplerian orbit can be written as:

$$E = \left(\frac{GMm}{L}\right)^2 \frac{m}{2}\left(e^2 - 1\right) \tag{5}$$

The full derivation of this is given in Ryden – I'll defer to the Chapter on Keplerian orbits in the book.

3. **Accretion disks** Consider a little packet of mass Δm in a Keplerian orbit around a central mass M (spherical, with radius R) at radius a_i.

(a) Write down an expression for the total orbital energy of the system. **Just Keplerian:** $E = -GMm/2a$**.**

(b) Now suppose the little mass packet Δm experiences a little "friction" in its orbit. Does a decrease or increase over time? **Friction takes energy OUT of the system, so E has to decrease.** $E \propto -1/a$ **so for energy to decrease a has to decrease.**

(c) Eventually the mass packet Δm falls onto the central object (recall the central object has radius R). Write down again an expression for the *change* in orbital energy ΔE between the initial position of the mass packet and final position. Also simplify your expression by assuming $a_i \gg R$. **The Keplerian orbital energy as the mass packet reaches the central object is** $E = -GM\Delta m/R$**. The change in energy is:**

$$\Delta E = -\frac{GM\Delta m}{R} + \frac{GM\Delta m}{a}. \tag{6}$$

But since we're told $a \gg R$, we can drop the second term (negligible) and we get $\Delta E \approx -\frac{GM\Delta m}{R}$**.**

(d) Now suppose I have not one, but *many* little mass packets falling in over a timespan Δt. What is the resulting orbital energy released per time: $L \approx \frac{\Delta E}{\Delta t} \approx \frac{GM\Delta M}{R\Delta t}$**. Note that** $\Delta m/\Delta t \equiv \dot{M}$ **is the "accretion rate" so that we can write:**

$$L \approx \frac{GM\dot{M}}{R}. \tag{7}$$

[1]For the aficianados: ignore all cosmological effects.

The released orbital energy has to go somewhere. Where does it go?? It ends up being released in the form of photons: this is known as the "accretion luminosity" – as a central object accretes matter, the matter releases orbital energy that can then shine! You will learn more about this in Astro 7B, but this ends up being the reason (paradoxically) that black holes are some of the brightest objects in the universe!

4. **Event horizon**

   (a) Consider a little mass Δm around a black hole of mass M, located at radius R (the mass is NOT in orbit, it is just sitting still!). Derive the radius of the event horizon of the black hole by setting the total energy of mass m equal to zero and setting $v = c$. $mv^2/2 = GMm/r \rightarrow R_{sch} = 2GM/c^2$.