Stars 'n stuff

1. **Gravitational instability** Consider a large cloud of gas, with mass M, radius R, and some temperature T.

   (a) Derive the free fall time $t_{\rm ff}$. Write in TWO forms – one in terms of only ρ (+ fundamental constants) and another in terms of M, R, and fundamental constants. Bonus points if you can derive it in more than one way[1]. What does the free fall time mean, physically? **Recall that $t_{\rm ff}$ is the "characteristic timescale" to cross the system when ONLY GRAVITY acts. In this case. If we want to cross a system with radius R, and are accelerating solely due to gravity, then standard kinematics equations tell us:**

   $$R = \frac{1}{2}gt^2 \rightarrow t \sim \sqrt{gR}, \tag{1}$$

   where g is the acceleration due to gravity and I have assumed that it is a constant. As we are crossing the system, is there any reason to believe that the acceleration due to gravity is ACTUALLY constant? No, not really! Remember $g \equiv GM(<r)/r^2$ and both $M(<r)$ and r are changing as we move inward. But we are going to ignore these annoying complications and pretend the acceleration is constant. For simplicity, we'll evaluate g at the surface of the object, so that we just have $g = GM/R^2$. Plug in and you'll find:

   $$t_{\rm ff} \sim \frac{1}{\sqrt{G\rho}}. \tag{2}$$

   Use $\rho \sim M/R^3$ to get it in terms of M and R if you like. As a sanity check, what happens to $t_{\rm ff}$ if ρ increases? Convince yourself that the direction makes sense.

   (b) Derive the sound crossing time $t_{\rm sound}$ in terms of T, $\bar{m}$, and the object's size R. **How fast does sound cross the system? Well, $d = vt$ so if $d = R$ then:**

   $$t_{\rm sound} \sim \frac{R}{v_{\rm sound}}. \tag{3}$$

   What is $v_{\rm sound}$? This is the trickiest part here, because we have to understand what "sound" is. Sound waves are just disturbances in pressure/density that propogate throughout a gas. If there is a "disturbance" somewhere (suppose someone sticks their hand in and swirls the gas around a bit), then information about this disturbance travels across the gas at the speed of the sound. We never actually derived the sound speed for you (take a fluid dynamics class for the whole nitty-gritty derivation). But, we know that sound waves are associated with gas. Recall that using the Maxwell Boltzmann distribution we derived the typical speed of a PARTICLE in a gas is:

   $$v \sim \sqrt{kT/\bar{m}}. \tag{4}$$

[1] Except this is only for your benefit, so its only bonus points in your heart. Anyways, I often find that when I know how to derive something using multiple different routes that I understand it more deeply.

Now, a sound wave is not quite the same as a particle in the gas, but these are both phenomena associated with pressure. So I hope you will not throw a tomato at me if I say that:

$$v \sim \sqrt{kT/\bar{m}}. \tag{5}$$

In other words, sound waves propagate across the gas at roughly (modulo order-unity factors) the same speed as a typical particle in the gas. So, we have:

$$t_{\text{sound}} \sim \frac{R}{\sqrt{kT/\bar{m}}}. \tag{6}$$

By the way, you can derive t_{ff} in an analogous fashion to what we did here, i.e.:

$$t_{\text{ff}} \sim R/v_{\text{grav}}, \tag{7}$$

where v_{grav} is the typical speed associated with GRAVITY ONLY. Can you guess what a characteristic speed for gravity is?

(c) For gravitational instability to set in, i.e. for the cloud to collapse under its own weight, who has to win – pressure or gravity? Translate this into a criterion on the ratio $t_{\text{sound}}/t_{\text{ff}}$ for gravitational instability. **Gravity has to win. For gravity to win, the time for pressure to STABILIZE the system has to be shorter (in other words, the sound waves travel too slowly to respond to perturbations due to gravity) than the timescale for gravity to collapse the cloud. In other words:**

$$t_{ff} < t_{\text{sound}}. \tag{8}$$

(d) Use this criterion to derive an expression for the Jeans mass in terms of ρ and T. Don't worry about order-unity constants.

Use the expressiosns for t_{ff} and t_{sound}, set them equal, and solve for M to find:

$$M_J \propto T^{3/2}\rho^{-1/2} \tag{9}$$

(e) Does M_J increase or decrease as ρ increases? Same question for T. Does the direction make sense? **As ρ increases M_J DECREASES – this means it becomes EASIER to collapse the cloud (you don't need to put as much mass on your cloud before it becomes unstable to gravity). This make sense – ρ is a measure of how strong gravity is. As T increases M_J INCREASES – this means it becomes HARDER to collapse the cloud. Again, this make sense: T is a measure of the thermal pressure of the cloud (hotter cloud = particles in the gas flying about faster = harder to collapse). So if the temperature is higher, then pressure is stronger.**

2. **Protostellar contraction** Our cloud from above is now collapsing under its own self-gravity.

(a) Does its total energy of the cloud INCREASE or DECREASE? (You can use the virial theorem to answer this part and the next part; *hint*: for once, signs actually matter) **Recall the virial theorem says that $E = K + U = -U/2$. Hopefully I didn't mess up a factor of two somewhere. Anyways, since $E \sim -U$, and we know $U \sim -GM^2/R$, we know:**

$$E \sim -\frac{GM^2}{R}. \tag{10}$$

Since R is decreasing and M stays constant, E DECREASES because it becomes more negative (though the magnitude of E is increasing).

(b) Likewise, does the kinetic energy (read: temperature) in the gas cloud INCREASE or DECREASE? **By the virial theorem $E = K + U = -2K$, so $dE = -2dK$. Since $dE < 0$, we have that $dK > 0$. The kinetic energy INCREASES, so the cloud gets hotter (this should make sense intuitively).**

(c) In part (a) above, you should have found that the total energy of the cloud is decreasing. Where does the energy go? **Some of the energy is going to into heating up the cloud. But not all (only half of it goes to heating the gas, by the VT). The other half is radiated away, in the form of photons. This is why contracting protostars are bright.**

(d) Write a differential equation relating the luminosity of the cloud L to the rate of contraction dR/dt. Do not worry about order unity constants or actually solving the equation. This is just a quantitative way of framing part (c) above. **The luminosity comes from the change in energy. So:**

$$L = \frac{dE}{dt} = \frac{d}{dt}\frac{GM^2}{R} = \frac{GM^2}{R^2}\frac{dR}{dt}. \tag{11}$$

What this says is that the luminosity (energy per time) depends on how fast the cloud is contracting.

(e) Provide an order of magnitude estimate of how long it will take the cloud to contract to its final state if the cloud has mass $M_\odot$. Does your answer depend on the initial cloud size? Why or why not? Would the contraction happen faster or slower if $M > M_\odot$? **The cloud contracts on a Kelvin-Helmholtz timescale:**

$$t_{\mathrm{KH}} \sim \frac{\Delta E}{\dot{E}} \sim \frac{\Delta E}{L}. \tag{12}$$

At the level of this class, we don't have an independent way of getting L. So for simplicity, we'll take the ENDPOINT of the evolution, at which point we do know the luminosity. And what is the endpoint but the Sun itself, for which $L = L_\odot$? Here ΔE is the CHANGE in energy. Recall:

$$\Delta E = E_f - E_{\mathrm{init}}. \tag{13}$$

But $E \sim -GM^2/R$ and $R_f \gg R_{\mathrm{init}}$ so $\Delta E \sim E_f \sim -\frac{GM_\odot^2}{R_\odot^2}$. So the answer does NOT depend on the initial cloud size!

3. **A star is born**

   (a) The protostar's collapse halts when fusion ignites in the center. Derive an order-of-magnitude expression for the requisite temperature T_{fuse} at the center of the protostar to ignite hydrogen burning.

   Quantum tunneling can happen when the (classical) distance of closest approach r_c between two protons whizzing about in the Sun is of order the de-Broglie wavelength of the particles. We can get a handle

   $$\frac{e^2}{r_c} \sim k_B T. \tag{14}$$

   And the de Broglie wavelength is $\lambda \sim \hbar/p \sim \hbar/mv_{\text{th}}$, where v_{th} is the typical thermal velocity.

   (b) After the onset of fusion who is winning, pressure or gravity? **Neither**

4. **Under pressure**

   (a) Derive an order of magnitude expression for the central pressure of a self-gravitating object in hydrostatic equilibrium in terms of mass M, radius R, and fundamental constants. $P_c \sim \frac{GM^2}{R^4}$

   (b) What holds up MOST stars on the main sequence? That is, what is the SOURCE of P_c you calculated in the above part? In yet other words, what is counteracting gravity[2]? Write the usual expression for it. **gas pressure** $P = \rho k_B T/\bar{m}$

   (c) What holds up the most massive stars on the main sequence? Write an expression for this kind of pressure. **[after lecture tomorrow]** **radiation pressure $P \sim aT^4$ where a is the radiation constant.**

5. **It's getting hot in here.**

   (a) Now, let's further assume that the central pressure is due to your answer in part 4(b). Set the pressures equal to derive an expression for the central temperature T_c in terms of only M, R, $\bar{m}$, and fundamental constants. $T_c \sim GM\bar{m}k_B R$

   (b) Show that you can get the same expression as above by applying the virial theorem (it's the same physics, in disguise!). **VT says that the kinetic energy is of order the potential energy. Therefore $k_B T_C \sim \frac{GM\bar{m}}{R}$ which is the same as what we have above.**

   (c) Evaluate your expression for T_c for the Sun. Compare to your answer in 3(a). Do the numbers check out?

   Should get $T_c \sim 10^7$ K for both.

   (d) Repeat part (a) but now assume that the central pressure is due to your answer in part 4(c) **[after lecture tomorrow]**

6. **Fuel in the tank**

   (a) Estimate how long before a 1 $M_\odot$ uses up all its hydrogen. Assume that only 10% of the star's hydrogen can actually be fused[3]. Assume that p-p fusion dominates.

[2] Somewhat confusingly, even though collapse stops when fusion ignites, the star is NOT supported by radiation pressure, it is supported by gas pressure. The energy from the photons thermalizes the gas, heating it up. This increases the pressure in the gas – but the pressure is in the GAS not in the photons, at least for most stars (massive stars, on the other hand, ARE dominated by radiation pressure).

[3] Why not 100%, you ask? Think about what the temperature profile of the star looks like.

You will need to calculate the energy liberated in a p-p reaction. I always find it very helpful to remember that the efficiency of the reaction is 0.7%. That is, you release 0.7% of the rest mass energy you started with.

We get $E_{\rm rxn} \sim .007(4m_p c^2)$ of energy per reaction. The total number of fusion reactions possible is $N_{\rm rxn} \sim \frac{M_\odot}{4m_p}$. Plug in numbers and you'll get:

$$t_{\rm MS,\odot} \sim 10{\rm Gyr}. \tag{15}$$

(b) Now assume that $L \propto M^\alpha$, where α is some constant "power-law index". Scale your answer above to write the main sequence lifetime $t_{\rm ms}$ as a function of stellar mass M, scaling off the Sun (i.e., write your answer in the form $t_{\rm MS} = K(M/M_\odot)^\beta$, where K is the main sequence lifetime for $M = M_\odot$ and β is a power-law index that you should determine (hint: it is related to, but not the same as, α).

$$t_{\rm MS} \propto \frac{M}{L} \propto \frac{M}{M^\alpha} \propto M^{1-\alpha}. \tag{16}$$

(c) If $\alpha > 1$, who dies first, the massive or low mass stars? What about $\alpha = 1$? What about $\alpha < 1$? In reality, the index α is itself a function of mass (in other words, the scaling between mass and luminosity changes as a function of mass). For stars around the mass of the Sun, $\alpha \approx 3.5$ (so $L \propto M^{3.5}$), whereas for high mass stars $\alpha \approx 1$. **If $\alpha = 1$, then $t_{\rm MS}$ is constant. If $\alpha > 1$, massive stars die first and if $\alpha < 1$, low-mass stars die first.**

(d) Humans are estimated to have been on earth for $\sim 10^5$ yrs. By what factor would you have to increase the mass of the Sun for its MS lifetime to be comparable to the existence of humanity? **The real main sequence lifetime is 10 Gyr $= 10^{10}$ yr. We need the MS lifetime to decrease by a factor of $10^{10}/10^5 = 10^5$. Let's take $\alpha = 4$. Then $t'_{\rm MS}/t_{\rm MS} = 10^{-5} = (M/M')^3 \to M'/M = (10^5)^{1/3} = 46$.**