1 Gravitational Focusing

In lecture, Eugene derived an expression for the collision rate between two types of particles. In his derivation, we accounted only for physical collisions, ignoring the fact that particles can also “collide” gravitationally. That is, their trajectories can be significantly deflected by gravitational attraction, approximating a physical collision. This is called gravitational focusing. Here we will derive a cross section that accounts for this effect.

a) Consider two identical particles (labeled 1 and 2) of mass $M$ and radius $R$. 1 passes by 2 with impact parameter $b$ and velocity $v_0$. Estimate the change in velocity, $\Delta v$, of 1 as it is deflected from its original trajectory due to the gravitational pull of 2.

Assuming that the effect on the trajectory of particle 1 occurs in a distance of $2b$, which should be familiar from lecture, we have a time of action given by

$$\Delta t = \frac{2b}{v_0}$$

where we have simply used dimensional analysis or the fact that velocity is distance over time. The acceleration during this time interval is approximately constant with a value of

$$a = \frac{GM}{b^2}$$

so that the change in velocity is given by

$$\Delta v = a \Delta t$$

$$\Delta v = \frac{2GM}{bv_0}$$

b) Gravitational focusing becomes significant when the change in velocity is on order-of-magnitude equal to the initial velocity (i.e. effect on order unity, $\Delta v \approx v_0$). How large does the impact parameter $b$ need to be for this to occur?

$$\Delta v = v_0$$

$$\frac{2GM}{bv_0} = v_0 \Rightarrow b = \frac{2GM}{v_0^2}$$

c) Using your answer to b), calculate the cross section, $\sigma_{gf}$, for gravitational focusing, in terms of $v_0$, $v_{esc}$, and $\sigma_{old}$, where $v_{esc}$ is the escape velocity of 1’s gravitational field, and $\sigma_{old}$ is the interaction cross section we derived in class, repeated in Equation 1 for convenience.

$$\sigma_{old} = \pi(R_1 + R_2)^2$$
For the new cross-section, the effective radius is \( b \), which should logically make sense. Thus, we have

\[
\sigma_{gf} = \pi b^2
\]

and can plug in our expression for \( b \) from part b) to get

\[
\sigma_{gf} = \pi \left( \frac{2GM}{v_0^2} \right)^2
\]

To put this in terms of \( v_{esc} \) and \( \sigma_{old} \), we recognize that \( v_{esc}^2 = 2GM/R \). Thus,

\[
\sigma_{gf} = \pi \left( \frac{v_{esc}R}{v_0^2} \right)^2 = \left( \frac{v_{esc}}{v_0} \right)^4 \pi R^2
\]

Since the old cross section is given by Equation 1 with \( R_1 = R_2 = R \), we have \( \sigma_{old} = 4\pi R^2 \). Substituting this into our expression,

\[
\sigma_{gf} = \frac{1}{4} \left( \frac{v_{esc}}{v_0} \right)^4 \sigma_{old}
\]

d) How does \( \sigma_{gf} \) compare to \( \sigma_{old} \)? Do a sanity check: does this relationship make sense, intuitively? Why or why not?

As long as \( v_0 < v_{esc} \), \( \sigma_{gf} > \sigma_{old} \). When the particle velocity is slow compared to the strength of the gravitational acceleration it feels, gravitational focusing provides a larger cross-section than direct collision. When it moves very fast compared to the gravitational acceleration it feels, the gravitational focusing cross-section is less than that of direct collision, indicating that the two particles are near each other for so short a time that the effect is smaller, which makes sense.

### 2 Jeans Instability

First, a refresher. Recall that Equation 2 describes the Jean’s length, and whether the radius is greater than or less than this length determines whether an interstellar cloud will collapse.

\[
R_J \approx \sqrt{\frac{k_BT}{\mu m_H G \rho}}
\]  

(2)

Jean’s mass is rewritten from the lecture notes in Equation 3.

\[
M_J \approx \frac{k_B^{3/2}T^{3/2}}{(G\mu m_H)^{3/2} \rho^{1/2}}
\]  

(3)

You should review your notes to make sure you understand the derivations of these quantities, but I provide them for you here for convenience. You can use \( \mu m_H \approx 1.67 \times 10^{-27} \text{ kg} \).
a) **Concept Check.**

(a) As a star collapses, how does its temperature and density change? What does this tell you about how the Jeans radius and Jeans mass change during collapse?

The density and temperature both increase. Both Jeans radius and Jeans mass vary directly with temperature and inversely with density. Thus, it is clear that whether these quantities decrease or increase as the star collapses depends on the particular system and is not immediately obvious. You will work this out in your problem set this week.

(b) Which two velocities are balanced at Jeans instability? In order for collapse to occur, which one must be greater than the other?

The thermal and escape velocities are approximately balanced at the border of Jeans instability. For collapse to occur, escape velocity must be greater than thermal velocity. If you ever forget this, consider logically: for things to form, you need stuff to slow down for long enough to come together. Therefore, greater $v_{thermal}$ means things are moving too fast to chill out and condense into a star. The comparison to escape velocity should make sense, since if the gases are moving quickly enough to overcome the gravitational force, they can’t come together into gravitational collapse. In math,

$$v_{thermal} < v_{esc} \Rightarrow \text{collapse}$$

b) Consider the acceleration felt by a particle at the edge of the Jean’s cloud and use a basic kinematic equation relating time, distance, and acceleration to derive the characteristic free-fall time for a Jean’s cloud with mass density $\rho$ to collapse. Note: your answer should depend only on $\rho$ and constants.

The acceleration felt by a particle at the edge of a Jean’s cloud with mass $M$ and radius $R$ is just

$$a = \frac{GM}{R^2}$$

and the basic kinematic equation we want to use is

$$d = \frac{1}{2}at^2,$$

which gives distance travelled due to acceleration $a$ over a time $t$. Solving for time,

$$t = \sqrt{\frac{2d}{a}}$$

and plugging in for acceleration, the free-fall time is

$$t_{ff} = \sqrt{\frac{2R^3}{GM}}$$

To rewrite this in terms of only density, note that

$$\rho = \frac{M}{V} = \frac{M}{4\pi R^3/3}$$
Therefore, the quantity in the square root can be replaced and we have

\[ t_{ff} = \sqrt{\frac{3}{2\pi G\rho}} \]

in terms of only constants and the density!

c) Typical interstellar molecular clouds have number densities \( n_H \approx 10^3 - 10^4 \) atoms/cm\(^3\) and temperatures of \( T \approx 30 \) K. At what minimum radii \( (R_j) \) will the clouds collapse? Hint: You will need to rewrite Equation 2 in terms of \( n_H \) instead of \( \rho \) (use dimensional analysis).

We are given \( n_H \) but the equation I provided for you has \( \rho \), so we need to realize \( n_H \) has units of atoms per volume while \( \rho \) has units of mass per volume. Thus, \( \rho = n_H \times \mu m_H \) is the only logical substitution. From here, it’s just a matter of plugging into

\[ R_J = \sqrt{\frac{k_B T}{G(\mu m_H)^2 n_H}} \]

to get numerical answers. Mine are:

\[ R_J \approx 0.48 - 1.53 \text{ pc} \]

where the range comes from plugging in \( n_H = 10^3 \) or \( n_H = 10^4 \) atoms/cm\(^3\) and the larger radius comes from the lower density. This makes sense since you’d expect less dense clouds to have to be larger in order to have enough gravitational strength to force collapse, whereas a small dense cloud should collapse more easily.

d) What are Jean’s masses (in solar masses) of the clouds described in the part a)?

Again, we just plug into Equation 3 while making the simple substitution \( \rho = n_H \mu m_H \) to get

\[ M_J = \frac{(k_B T)^{3/2}}{G^{3/2}(\mu m_H)^2 n_H^{1/2}} \]

Numerically, I found

\[ M_J = (5.54 - 17.5) \times 10^{31} \text{ kg} = 28 - 90 \text{ } M_{\odot} \]

where, again, the larger quantity in the range comes from the lower density, as expected.

e) A Jean’s cloud has a number density \( n_H = 2 \times 10^3 \) cm\(^{-3}\). How long will it take the cloud to collapse (ignoring rotation and magnetic fields)? Hint: use your answer from part b).

This is a plug and chug but having a sense of the free-fall times for collapse is generally useful. We plug our values into

\[ t_{ff} = \sqrt{\frac{3}{2\pi G\mu m_H n_H}} \]

and the numerical answer I obtained is

\[ t_{ff} = 1.47 \times 10^6 \text{ years} \]