Measuring Nuclear Spin and Magnetic Fields with Optical Pumping

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ABSTRACT

We use the technique known as optical pumping to measure the nuclear spin of Rb^{87} and Rb^{85} as well as the strength of the magnetic field of the Earth. We use plots of the resonance frequency versus the current or magnetic field from a Helmholtz coil for each isotope, the Breit-Rabi equation, and the expression for the magnetic field strength at the center of a Helmholtz coil to find a nuclear spin value of 3/2 and 5/2 for Rb^{87} and Rb^{85}, respectively, and 0.316 ± 0.004 gauss for the Earth's magnetic field strength.

1. Introduction

1.1. The Atom and its Energy Levels

The atom consists of a dense central nucleus composed of protons and neutrons with electrons occupying orbitals around the nucleus. Each orbital is labeled by a set of quantum numbers and has associated with it a specific energy. If orbitals have the same energy they are called degenerate.

The Coulombic interaction between the negatively-charged electron and the positively-charged nucleus gives rise to the principle energy levels of the atom. The coupling of the electron's orbital angular momentum $\mathbf{L}$ and its spin angular momentum $\mathbf{S}$ is responsible for the fine structure of atoms. The fine structure sublevels are also split into more sublevels due to the interaction of the nuclear spin $\mathbf{I}$ and the total electronic angular momentum $\mathbf{J}$. This is known as the hyperfine structure of the atom. If the atom is placed in an external magnetic field, the Hamiltonian of the system depends on the orientation of the total angular momentum of the atom $\mathbf{F}$ with respect to the external magnetic field. Thus, each hyperfine level is split in what is known as the Zeeman effect.

1.2. Rubidium Energy Levels

The rubidium atom has one electron in its outer valence shell. In the ground state, this electron has $S = 1/2$ and $L = 0$. By the rules for the addition of angular momenta, the total electronic

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angular momentum is $J = 1/2$. In spectroscopic notation, the ground state of rubidium is ${}^2S_{1/2}$. In the first excited state we have $S = 1/2$ and $L = 1$ so $J = 3/2, 1/2$. Thus, the first excited state is split into two different energy levels ${}^2P_{3/2}$ and ${}^2P_{1/2}$ by spin-orbit coupling.

Rb^{87} has a nuclear spin of $I = 3/2$. In its ground state ${}^2S_{1/2}$, $J = 1/2$ so there are two values for the total angular momentum of the atom, $F = 2, 1$. This is also true for the excited state ${}^2P_{1/2}$ since $J = 1/2$ for this state. For the excited state ${}^2P_{3/2}$, $J = 3/2$ so $F = 3, 2, 1, 0$. Thus, the hyperfine interaction produces two different energy levels for each $J = 1/2$ state and four different energy levels for the $J = 3/2$ state.

If we place the Rb^{87} atoms in a weak external magnetic field, each F state will be split into more substates labeled by the quantum number m_F. For a given value of F, $m_F = +F, F-1, \ldots, -F$. Thus, the Zeeman splitting produces $2F+1$ different energy levels for each F state. For example, in the ${}^2S_{1/2}$ ground state with $F = 1$, there are 3 total Zeeman levels $m_F = 1, 0, -1$. In an attachment we show an energy level diagram for Rb^{87} and Rb^{85}. These are not drawn to scale.

1.3. Optical Pumping

When a system is in thermal equilibrium, atoms are most likely found in the ground state configuration. Optical pumping is a technique used to produce a population inversion. We place Rb atoms in a external magnetic field and bombard them with circularly-polarized light that corresponds to the transition from ${}^2S_{1/2}$ to ${}^2P_{1/2}$. Since the light is polarized and photons carry 1 unit of angular momentum, the m_F value of the electron must increase by 1 when it transitions to the excited state. When the electron finally drops back down, we must have $\Delta m_F = 0, \pm 1$ by the selection rules. Thus, the electron has a $2/3$ probability of having a higher m_F value when it drops back down to the ground state than before. In Rb^{87}, all of the atoms will eventually reach the ${}^2S_{1/2}$ state with $F = 2$ and $m_F = +2$. In Rb^{85}, all of the atoms will eventually reach the ${}^2S_{1/2}$ state with $F = 3$ and $m_F = +3$. When electrons reach these states, the atoms cannot absorb any more of the polarized light because there is no longer a transition that increase the m_F value of the electron by exactly 1. Thus, we say that the electrons have been “pumped” into a certain configuration.

1.4. Measuring Nuclear Spin and Earth’s B-field

In our experiment, we will optically pump Rb atoms to measure the nuclear spin of Rb^{87} and Rb^{85}. The strength of the external magnetic field determines the difference in energy of each Zeeman level. This relationship depends also on the nuclear spin of the atom and is given by the Breit-Rabi equation

$$\frac{\nu}{B_{ext}} = \frac{2.799}{2I+1}\ \text{MHz gauss}^{-1} \tag{1}$$

where B_{ext} consists of the magnetic field of the Earth (B_E) and that from a Helmholtz coil (B_H) and ν is the frequency of a photon whose energy corresponds to the Zeeman transition.

To derive Eq. (1), we begin with the Hamiltonian for our atom

$$H = \hbar I(\mathbf{I}\cdot\mathbf{J}) - \boldsymbol{\mu}_J\cdot\mathbf{B}_{ext} - \boldsymbol{\mu}_I\cdot\mathbf{B}_J - \boldsymbol{\mu}_I\cdot\mathbf{B}_{ext} \tag{2}$$

The third term on the RHS is very small because B_J is very small compared to other effects. If the external magnetic field is weak, the first term dominates and only components along $\mathbf{F}$ need to be considered. Since both $\mathbf{B}_{ext}$ and $\boldsymbol{\mu}_I$ are small along the direction of $\mathbf{F}$, we can ignore it as well. Thus, our Hamiltonian becomes

$$H = \hbar I(\mathbf{I}\cdot\mathbf{J}) - \mu_J\frac{\mathbf{J}\cdot\mathbf{F}}{F}\frac{\mathbf{F}}{F}\cdot\mathbf{B}_{ext} \tag{3}$$

Then

$$\langle H\rangle = \frac{\hbar I}{2}[F(F+1)-I(I+1)-J(J+1)] - \frac{\mu_J}{2}\frac{[F(F+1)+J(J+1)-I(I+1)]}{F(F+1)}mB_{ext} \tag{4}$$

For a transition in Rb with $J = 1/2$ from an m state to an $m+1$ state we have

$$E_{m+1} - E_m = h\nu = \frac{\mu_J B_{ext}}{2}\frac{[F(F+1)+3/4-I(I+1)]}{F(F+1)} \tag{5}$$

If we substitute $F = I + J = I + 1/2$ and simplify we get

$$h\nu = \frac{\mu_J B_{ext}}{2I+1} \qquad \Leftrightarrow \qquad \frac{\nu}{B_{ext}} = \frac{\mu_J}{h}\frac{1}{2I+1} \tag{6}$$

We use a light detector to determine the level of pumping. Initially, the Rb sample is opaque because the atoms absorb all the optical light incident upon it. The atoms gradually reach the pumped state at which time they cannot absorb anymore light and the sample becomes transparent. If we induce the atoms to drop down one m_F state using an RF signal, the sample can absorb light once again and less light is transmitted through the sample. For a given B_{ext}, the RF frequency that produces the minimum light transmission is the frequency of the Zeeman transition. We measure the resonance frequency for several values of the B_{ext} for each isotope and determine their nuclear spin. With the nuclear spin values, we can turn off the B_H and find the resonance frequency for the magnetic field of the Earth. This will allow us to determine the strength B_E.

2. Experimental Setup

We show a block diagram of the experimental setup in an attachment to this report. We discuss the important components in turn.

2.1. Helmholtz Coil

We have a Helmholtz coil with a radius $R = 27.5$ cm and $N = 135$ turns of the coil. The simplest model of a Helmholtz coil consists of two circular loops separated by a distance d with the same current i flowing in the same direction in each loop. To find an expression for the B-field of this Helmholtz coil we first consider the B-field from a single wire loop lying in the xy-plane at $z = 0$. If i is in the counternclockwise direction when looking down the z-axis, then the B-field on the z-axis at a distance z above the circular loop is

$$\mathbf{B}(z) = \frac{\mu_0 \cdot i}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{\mathbf{z}}$$

Now if we place two identical loops at $z = \pm d/2$. Then B-field from the two loops is given by the principle of superposition

$$\mathbf{B}(z) = \frac{\mu_0 i R^2}{2} \left\{ \frac{1}{[R^2 + (d/2 + z)^2]^{3/2}} + \frac{1}{[R^2 + (d/2 - z)^2]^{3/2}} \right\} \hat{\mathbf{z}} \tag{7}$$

It can be shown that $(\partial B/\partial z) = 0$ at z=0 for Eq. (7). In addition, if we pick $d = R$, $(\partial^2 B/\partial z^2) = 0$ at $z = 0$ and the magnetic field strength is

$$\mathbf{B}(0) = \frac{8\mu_0 i}{5^{3/2} R} \hat{\mathbf{z}} \tag{8}$$

If each circular wire were instead a coil of N turns, by the principle of superposition we would in essence have N circular wires and the B-field at the center would be be N times Eq. (8). If we substitute $\mu_0 = 4\pi \times 10^{-7}$ N A^{-2}, Eq. (8) becomes

$$B_H = 0.9 \times 10^{-6} \cdot \frac{Ni}{R} \text{ tesla m A}^{-1} \tag{9}$$

We vary B_H by adjusting i inside the coils. This is supplied by a DC power supply. The DC power supply is also connected to an AC modulation unit that allows us to modulate the B-field around some DC value. Reasons for doing this will be explained later.

2.2. Rubidium Light Box

Placed at the center of the Helmholtz coil is the Rb light box. This is where all the physics takes place. Inside the Rb light box there is an Rb bulb, an Rb lamp and circular polarizer, a heater coil, and an RF coil. The Rb lamp produces optical light that corresponds to the D1 transition in Rb. The light goes through the circular polarizer before hitting the Rb bulb. The heater coil is connected to a heater control unit that we use to heat the sample and monitor the temperature inside the box. A DS 345 function generator is connected to the RF coil and is used to produce the

RF signal that induces the Rb electrons to drop down one m_F state once they have been pumped. Finally, a photodiode detector is connected to the Rb light box to monitor the opacity of the Rb bulb.

3. Experimental Procedure

3.1. Measuring Resonance Using Frequency Modulation

Before we measure the resonance frequency for each isotope at different i values, we explore how to determine when the resonance condition has been established. The first method of establishing resonance is to keep i fixed and vary the frequency of the RF signal.

We place the rubidium bulb into the Rb light box and heat the sample to about 45 °C. We set the DC power supply to a current of 1.0 A and keep the field modulation unit off. The DS 345 is turned on and set for frequency modulation (FM) of a 3 MHz sine wave with a 10 Hz ramp function. The ramp function causes the frequency of the sine wave to increase linearly with time during one period. The signal is sent to the RF coil and CH.1 (X) of the oscilloscope. The photodiode signal is sent to the pre-amp and then to CH.2 (Y) of the oscilloscope. We set the oscilloscope in X-Y mode. On the oscilloscope, we see the variation in the opacity of the sample as a function of frequency.

To get a more accurate measurement of the frequency of the resonance peaks, we adjust the frequency of the sine wave and decrease the span of the modulation. Doing this we find an approximate resonance frequency for $i = 1.0$ A to be 3.318 MHz for Rb^{87} and 2.212 MHz for Rb^{85}.

Before moving on, we vary some of the parameters to get a feel for how they affect the resonance frequency. The temperature changes the size of the resonance peak but not the actual frequency. From the appendix of the Optical Pumping Write-Up, we see that the optical signal is a function of the temperature. Changing i clearly changes the resonance frequency. i controls B_H, which determines the energy difference between the Zeeman splitting. If we reverse the polarity of B_H, we can find a value for i at which there is no resonance. This occurs when B_H cancels B_E and there is no external magnetic field and all the Zeeman levels are degenerate.

3.2. Modulation of the Magnetic Field

The second method to measure the resonance frequency involves modulation of the B_H instead of modulating the frequency of the RF signal. Using B_H modulation we can either determine the resonance frequency with the oscilloscope in time trace mode or X-Y mode.

3.2.1. Time Trace Mode

We turn off the modulation on the DS 345 and turn on the field modulation. The field modulation is at 60 Hz because this is the frequency of the AC signal in the PG&E power line. We send the field modulation signal through a divid-by-ten attenuator and connect it to CH.1 (X) of the oscilloscope. We leave the amplified photodiode signal connected to CH.2 (Y). The scope is put into Dual Trace Mode with trigger on Line Source. We display CH.2 on the scope and observe the time variation of the opacity of the bulb as we vary i while leaving ν fixed. We change i until the pattern of light variation is a sine wave. If we are at the value of i that corresponds to resonance for a given ν, we expect changes in the opacity to be symmetric about this value of i. Hence, a sinusoidal variation of i will lead to a sinusoidal variation in the opacity of the bulb. Our uncertainty in the method is limited by fluctuations in i.

3.2.2. X-Y Mode

If we put the oscilloscope in X-Y mode, a Lissajou curve is displayed. There should be symmetry in the figure about the Y axis. The Y axis is displaying the variation of the opacity of the sample a function of i. If the DC i value is at resonance, we expect the variation in the opacity of the sample to by symmetric about this i value. We set the current at in the coil at 1.0 A and the modulation amplitude to 10. We change the frequency of the RF signal from the DS 345 and lower the modulation amplitude to find estimates for the resonance frequency of the two rubidium isotopes. We find that the resonance frequency for Rb^{87} and Rb^{85} is approximately 3.285 MHz and 2.171 MHz, respectively.

Again, before moving on, we experiment with the parameters when using B_H modulation to get a feel for our measurements. The current changes the symmetry of the Lissajou curve because it changes the resonance frequency. The measurement is quite sensitive to this change. In fact, it was one of our biggest problems. When making measurements, there were anomalous current fluctuations that caused our signal to be unstable. We know that the temperature affects the strength of the optical signal but it does not change the resonance frequency so our measurements are fairly insensitive to temperature changes within reason. The modulation phase only changes the symmetry of the Lissajou curve about the X axis and we only expect symmetry in the Y axis.

4. Pumping Time

We measure the pumping time of the Rb atoms at resonance using a square wave amplitude modulation. The field modulation is turned off and the coil current is set to 1.0 A. The DS 345 is set to modulate with depth of 100% at the resonance frequency and we send the modulated signal to CH.1. The square wave modulation has the effect of turning the RF signal on and off.

The photodiode detector signal on the oscilloscope resembles the charging and discharging of a capacitor. When the RF is on, the transmission increases sharply and levels off. Then when RF signal is off, the transmission of light decreases sharply and then levels off. This corresponds to atoms beginning to spontaneously fall down from the highest m_F state and the bulb becoming opaque. We adjust the modulation frequency so that we can see the pumping rate and relaxation rate level off significantly. By reading of the horizontal time scale of the oscilloscope, the estimated time constant for pumping is 50 μsec and 40 μsec for relaxation. The pumping time for the other isotope is different because there are different numbers of m_F states in the two. Since the selection rule dictates that the falling electron is 2/3 more likely to have a higher m_F, the more m_F values, the longer the pumping time. Along these lines it seems as though we would need to know the average time it takes an electron to spontaneously drop down in order to get an estimate of the pumping and relaxation time.

5. Measuring Resonance Frequency vs. Magnetic Field Strength

The heart of the optical pumping experiment is make a plot of resonance frequency versus current. From this data, we will be able to determine the nuclear spin of the two rubidium isotopes and obtain a value for B_E. We use the i modulation technique to determine resonance. The data for both rubidium isotopes is listed in Table 1. We use Eq. (9) to convert the values of i into B_H value.

5.1. Zero Field Resonance

We turn of the RF signal and adjust the current until with find a resonance value. This occurs when $i = 0.06$ A. There is a resonance value at zero frequency because the alignment of the Helmholtz coil field and the Earth's magnetic field is not perfect. Thus, we can use B_H to cancel out most of B_E, but there is still a small net field in an orthogonal direction.

6. Discussion

6.1. Nuclear Spin of $\mathbf{Rb^{87}}$ and $\mathbf{Rb^{85}}$

Our first task is to determine the nuclear spin of the two rubidium isotopes. We can write down the Breit-Rabi equation for both isotopes. If we take the ratio of ν_{87} and ν_{85} when B_{ext} is the same for both isotopes, it conveniently cancels out and we are left with a relation between I_{87} and I_{85}.

ν (MHz) Rb85	i (A)	B_H (gauss)	ν (MHz) Rb87	i (A)	B_H (gauss)
0.20	0.0150	0.0663	0.30	0.0187	0.08262
0.24	0.0381	0.1683	0.35	0.0357	0.1577
0.30	0.0702	0.3102	0.40	0.0526	0.2324
0.35	0.0953	0.4211	0.45	0.0702	0.3102
0.40	0.1199	0.5297	0.50	0.0859	0.3795
0.45	0.1455	0.6428	0.55	0.1035	0.4573
0.50	0.1699	0.7506	0.60	0.1198	0.5293
0.55	0.1940	0.8571	0.65	0.1370	0.6053
0.60	0.2201	0.9724	0.70	0.1540	0.6804
0.65	0.2440	1.0780	0.75	0.1705	0.7533
0.20	-0.1590	-0.7025	0.30	-0.1603	-0.7082
0.24	-0.1850	-0.8173	0.35	-0.1767	-0.7807
0.30	-0.2105	-0.9300	0.40	-0.1936	-0.8554
0.35	-0.2300	-1.0162	0.45	-0.2029	-0.8964
0.40	-0.2600	-1.1487	0.50	-0.2270	-1.0029
0.45	-0.2850	-1.2592	0.55	-0.2436	-1.0763
0.50	-0.3100	-1.3696	0.60	-0.2600	-1.1487
0.55	-0.3350	-1.4801	0.65	-0.2770	-1.2238
0.60	-0.3596	-1.5888	0.70	-0.2939	-1.2985
0.65	-0.3844	-1.6983	0.75	-0.3100	-1.3696

Table 1: The resonance frequency vs. B_H data for Rb^{87} and Rb^{85}. We modulate B_H and look for symmetry in the Lissajou curve about the Y axis of the oscilloscope.

$$\frac{\nu_{87}}{\nu_{85}} = \frac{2I_{85}+1}{2I_{87}+1} \tag{10}$$

Since the strength of the Earth's magnetic field is to a good approximation constant over our experiment time, we simply take the ratio of the resonance frequency for the two isotopes at the same value of i. We have one positive and one negative value of i for which the current is common for the two isotopes. For $i = 0.1198$ A, $\nu_{87} = 0.60$ MHz and $\nu_{85} = 0.40$ MHz. For $i = -0.2600$, $\nu_{87} = 0.60$ MHz and $\nu_{85} = 0.40$ MHz. These two sets of data points yield exactly the same ratio

$$\frac{\nu_{87}}{\nu_{85}} = \frac{3}{2} \tag{11}$$

This is exactly what we expect if the nuclear spin is 3/2 and 5/2 for Rb^{87} and Rb^{85}. However, I do not believe that the ratio in Eq. (10) and the expectation that the two spins are odd half-integral are enough to uniquely determine the two nuclear spins. For example, Eq. (11) is satisfied if the nuclear spin were instead 7/2 and 11/2 for Rb^{87} and Rb^{85}.

If we plug in $B_{ext} = B_H + B_E$ into Eq. (1) we get

$$\nu = \frac{2.799}{2I+1} B_H + \frac{2.799}{2I+1} B_E$$

We plot ν vs. B_H in Fig. (1) for both isotopes for both positive and negative currents. The slope of these curves should give us a measure for the nuclear spin of the two isotopes.

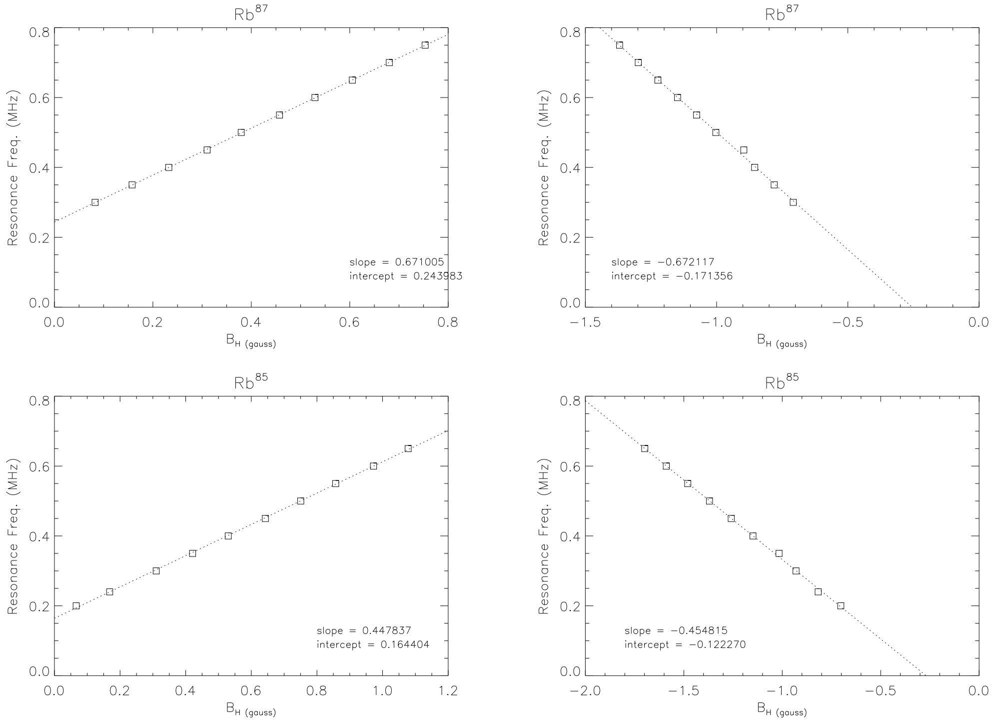

Fig. 1.— The resonance frequency ν versus the magnetic field strength of the Helmholtz coil. We use the slopes to get a measure of the nuclear spin of the two isotopes.

The slope of the Rb^{87} plots yield values of $I = 1.59, 1.58$ and the Rb^{85} plots yield $I = 2.61, 2.63$. If we combine these results with Eq. (11), we see that we must have $I_{87} = 3/2$ and $I_{85} = 5/2$. We do not bother stating an uncertainty because we know the value must be exactly half-integral.

6.2. Helmholtz Coil Field

Our expression for the field at the center of a Helmholtz coil in Eq. (9) was theoretically derived. It may not reflect the actual field at the Rb light box. Now that we have obtained the nuclear spin of the rubidium isotopes, we can use Eq. (1) to determine B_{ext} at the bulb. We do this for both isotopes for one positive and one negative current value. We compare this to the expected value for B_H using Eq. (9) with $R = 27.5$ cm and $N = 135$. The results are listed in Table 2.

	i (A)	ν (MHz)	B_{ext} (gauss)	B_H (gauss)
Rb^{85}	0.1199	0.40	0.8574	0.5297
Rb^{85}	-0.3100	0.50	1.0718	-1.3696
Rb^{87}	0.0187	0.30	0.4287	0.0826
Rb^{87}	-0.3100	0.75	1.0718	-1.3698

Table 2: After finding the nuclear spins of the rubidium isotopes, we are able to use the Breit-Rabi equation to find the strength of the external magnetic field at the Rb bulb. We compare this to the value of the Helmholtz coil field using Eq. (9).

The comparison of B_{ext} and B_H indicates the existence of B_E. We could use the data from Table 2 to get values for B_E. However, the values are not accurate because the value of B_H is not very actually since it was computed using Eq. (9). We can rewrite Eq. (9) as $B_H = \alpha i$. If we plug this into Eq. (1) we get

$$\nu = \frac{2.799}{2I+1}\alpha i + \frac{2.799}{2I+1}B_E \tag{12}$$

We plot the resonance frequency versus the current and write a program in IDL to fit the data to a line of the form $y = Ax + B$. The plots are shown in Fig. (2). Our program includes code to compute the error in the coefficients A and B. Here are the equations used in the fitting program.

$$A = \frac{\sum x^2 \sum y - \sum x \sum xy}{\Delta} \tag{13}$$

$$B = \frac{N\sum xy - \sum x \sum y}{\Delta} \tag{14}$$

$$\Delta = N\sum x^2 - (\sum x)^2 \tag{15}$$

$$\sigma_y = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}(y_i - A - Bx_i)^2} \tag{16}$$

$$\sigma_A = \sigma_y\sqrt{\frac{\sum x^2}{\Delta}} \tag{17}$$

$$\sigma_B = \sigma_y\sqrt{\frac{N}{\Delta}} \tag{18}$$

We use the slope to compute the constant α to be a better measure of B_H by

$$B = \frac{2.799}{2I+1}\alpha = \frac{2.799}{2I+1}\frac{N}{R}C \tag{19}$$

where C is the constant in front of Eq. (9) that we are interested in computing more accurately. Rb^{87} and Rb^{85} gives $C = (9.23 \pm 0.04) \times 10^{-7}$ and $C = (9.12 \pm 0.03) \times 10^{-7}$. The average of the two values yields $C = (9.18 \pm 0.04) \times 10^{-7}$.

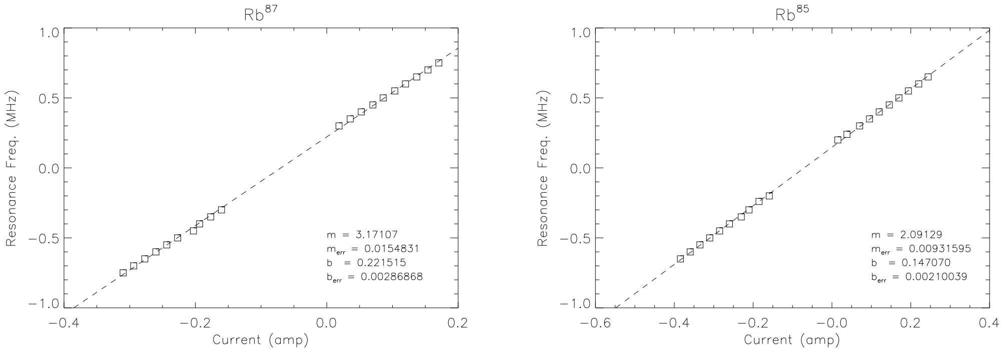

Fig. 2.— We the slope of the least squares line to compute the coefficient in Eq. (9) more accurately.

6.3. The Earth's Magnetic Field

We now use the coefficient we computed in the last section for the field from the Helmholtz coil to convert our values of i into values of B_H. We plot ν vs. B_H. The intercept of the least squares line will give us a value for B_E.

$$\nu = \frac{2.799}{2I+1} B_E \tag{20}$$

The plots are shown in Fig. (3). Rb^{87} and Rb^{85} yield $B_H = 0.315 \pm 0.004$ gauss and $B_H = 0.316 \pm 0.004$ gauss. We take the average as our value for the strength of the magnetic field of the Earth

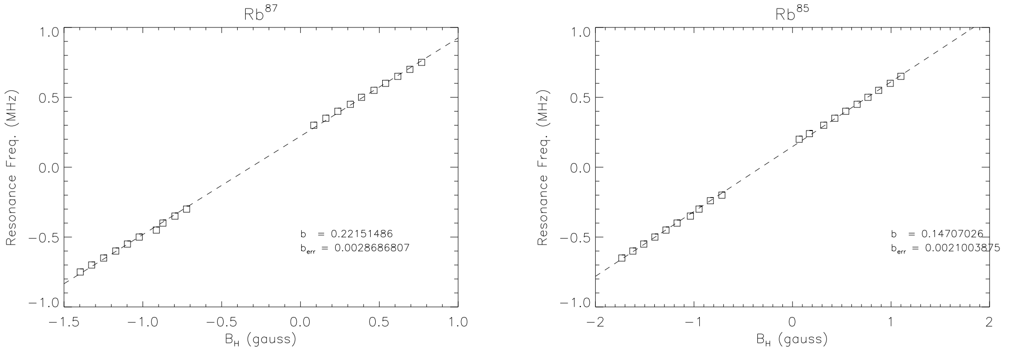

Fig. 3.— We use these plots of the resonance frequency versus the magnetic field of the Helmholtz coil to compute the magnetic field of the Earth from the intercept of the least squares line.

7. Conclusion

Our values for the Earth's magnetic field for the two isotopes are very close and within statistical error. It would be difficult to compare this to *the* value of B_E since it changes depending on the location of the observer. The range of B_E is less than 0.3 gauss to a little over 0.6 gauss. However, since the value we obtained from the two isotopes are so similar, we have confidence that we were able to successfully measure the strength of the Earth's magnetic field at our location. The plots of the resonance frequency versus either the current or Helmhotlz field yield slopes that agree with the expected nuclear spin of the two rubidium isotopes. We conclude that optical pumping is most certainly a useful technique in atomic physics to probe the structure of the atom.

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