ESSENTIALS OF SHOCKS, ASTRONOMY 127

Carl Heiles

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1. SHOCKS

1.1. Basic Physics

We treat shocks with the following assumptions:

1. Shocks are one-dimensional.

2. We develop equations in the rest frame of the shock.

3. We assume that the flow is semi-infinite, that is that the incoming (pre-shock, upstream) gas extends to infinity, as does the post-shock, downstream gas.

4. The shock is steady state: the flow has been going on forever and looks the same now as it always did.

5. We denote the upstream, pre-shock gas with subscript 1 and the post-shock gas with subscript 2.

6. The gas can by characterized by specific heats, with $p \propto \rho^\gamma$; $\gamma = 5/3$ for a monatomic gas, which is what we usually assume and for which numbers are given below.

7. The *isothermal* speed of sound is

$$c^2 = \frac{p}{\rho} \ . \tag{1}$$

The *adiabatic* speed of sound, which is the usual one because sound waves are much faster than heating/cooling, is $c_\gamma^2 = \frac{\gamma p}{\rho} = \gamma c^2$. These sound speeds can be expressed in terms of temperature, because $p = nkT = \rho kT/\mu$, where μ is the mean atomic weight (equal to 1.7×10^{-24} gm for a gas consisting of H atoms and 0.85×10^{-24} gm for ionized hydrogen gas):

$$c^2 = \frac{kT}{\mu} \ . \tag{2}$$

With these assumptions, all time derivatives are zero. We have three fundamental conservation relations that apply to the gas as it crosses the shock:

1.2. Conservation of Mass

The mass flow rates across unit area on each side are equal.

$$\rho_2 v_2 = \rho_1 v_1 \tag{3}$$

1.3. Conservation of Momentum

Newton's law: the pressure difference (force) is equal to the time rate of change of momentum.

$$p_2 + \rho_2 v_2^2 = p_1 + \rho_1 v_1^2 \tag{4}$$

Note that p and ρv^2 have the same units: pressure. The quantity ρv^2 is the pressure exerted by a blowing wind and is called the *ram pressure*. The ram pressure is less than the thermal pressure ($p = \rho c^2$) unless the wind is supersonic. It is often convenient to write the momentum equation

$$\rho_2(c_2^2 + v_2^2) = \rho_1(c_1^2 + v_1^2) \tag{5}$$

1.4. Conservation of Energy

The energy per unit mass on side 2 is equal to that on side 1, minus any energy that has been radiated in going across the boundary. The contributions to energy *per unit mass* are **(1) internal energy**, equal to $\frac{P}{(\gamma-1)\rho} = \frac{c^2}{(\gamma-1)} = \frac{3kT}{2\mu}$; **(2) PV energy** (the PdV work required to compress the gas), equal to $\frac{P}{\rho} = c^2 = \frac{kT}{\mu}$; **(3) kinetic energy**, equal to $\frac{v^2}{2}$.

The internal energy and the PV energy add together to give the *enthalpy* $H = \frac{\gamma P}{(\gamma-1)\rho} = \frac{5}{2}c^2$, which is the physically meaningful quantity for energy in this case.

The instantaneous energy loss rate per unit volume is $n^2\Lambda$, so the energy loss rate *per unit mass* is $\frac{n\Lambda}{\mu}$ and the total energy loss per unit mass is $\frac{1}{\mu}\int n\Lambda dt$. Note that $\frac{1}{\mu}\int n\Lambda dt$ can be rewritten by multiplying and dividing by nv and using the facts that (1) the path length $ds = vdt$ and (2) ρv is constant, yielding $\frac{1}{\mu}\int n\Lambda dt = \int n^2\Lambda ds$. Putting all this together we get

$$\frac{\gamma_2}{\gamma_2 - 1}c_2^2 + \frac{1}{2}v_2^2 = \frac{\gamma_1}{\gamma_1 - 1}c_1^2 + \frac{1}{2}v_1^2 - \int n^2\Lambda ds \tag{6}$$

With regard to conservation of energy, we have two types of shock: in the *nonradiative* or *adiabatic* shock, the radiative term $\int n^2\Lambda ds$ is negligible. In the *radiative* shock it is very important and cannot be neglected.

A special case of a radiative shock is an *isothermal* shock, in which the heating and cooling processes on each side are the same so that the temperatures on each side are the same. In this case the energy equation is just $T_2 = T_1$, or $c_2^2 = c_1^2$.

The loss term need not be positive: if it is negative, then there is energy gain and we have a *detonation front*. For example, if someone leaves the gas stove on and you walk in and strike a match, you generate a shock within which the oxygen and gas burn explosively and release energy in the front. Another example occurs in astronomy: the front that separates the ionized HII region gas from the neutral gas.

2. THE SHOCK EQUATIONS

2.1. Shocks of Arbitrary Strength

The solutions to the above equations (nonradiative case) are usually written in terms of the *Mach number* $\mathcal{M} = v_1/c_{\gamma,1} = v_1/\gamma^{1/2}c_1$. Note that $\mathcal{M}$ is defined relative to the *upstream, unshocked* gas; think of the case of a supersonic jet! The following solution assumes $\gamma_2 = \gamma_1$ and $\mu_2 = \mu_1$:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\mathcal{M}^2}{(\gamma-1)\mathcal{M}^2+2} \tag{7}$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\mathcal{M}^2 - \frac{\gamma-1}{\gamma+1} \tag{8}$$

$$\frac{T_2}{T_1} = \frac{p_2/p_1}{\rho_2/\rho_1} \tag{9}$$

If $\mathcal{M} \geq 1$, as is required for $\rho_2/\rho_1 > 1$, then we have a *shock*.

2.2. Strong Nonradiative Shocks.

In astronomy, we normally deal with HII regions and supernovae, which provide *large* pressures or energies: this is the case of a *strong shock*. In a strong shock, $\mathcal{M} \gg 1$; alternatively, $p_2 \gg p_1$. One can generate the solutions for a strong shock easily from equations 7 - 9 simply by considering the limit $\mathcal{M} \to \infty$.

However, there is a better way to picture a strong shock. Consider the momentum and energy equations 5 and 6. In a strong shock, $v_1^2 \gg c_1^2$. In other words, *the thermal pressure of the unshocked gas is negligible compared to its ram pressure*, and *the thermal energy of the unshocked gas is negligible compared to its kinetic energy.* For a strong shock, the right hand sides of the momentum and energy equations simplify because the terms involving c_1^2 can be neglected, and one can solve the equations rather more easily than in the general case.

The solutions for a strong nonradiative shock for the case $\gamma = 5/3$ are:

$$\frac{\rho_2}{\rho_1} = 4 \tag{10}$$

$$\frac{p_2}{p_1} = \frac{3}{4}\frac{v_1^2}{c_1^2}\ (\gg 1) \tag{11}$$

$$p_2 = \frac{3}{4}\rho_1 v_1^2 \tag{12}$$

$$\frac{T_2}{T_1} = \frac{3}{16}\frac{v_1^2}{c_1^2} \tag{13}$$

$$kT_2 = \frac{3}{16}\mu_2 v_1^2 \tag{14}$$

Note that p_2 and T_2, which express the energy content of the post-shock gas, depend *only* on v_1 and *not* on c_1; their expressions can be written *not* in terms of ratios involving c_1. This is a direct consequence of the "better way to picture a strong shock" mentioned in the above paragraph.

2.3. Strong Isothermal Shocks.

The solutions for a strong shock for the isothermal shock case can be generated from equations 7 - 9 by setting $\gamma = 1$, because in an isothermal shock $p \propto \rho$ and, with the definition of γ as $p \propto \rho^\gamma$, $\gamma = 1$ in an isothermal shock. Or, of course, one can solve the fundamental equations of mass, momentum, and energy for this special case. The solutions are:

$$\frac{\rho_2}{\rho_1} = \frac{v_1^2}{c_1^2}\ (\gg 1) \tag{15}$$

$$\frac{p_2}{p_1} = \frac{v_1^2}{c_1^2}\ (\gg 1) \tag{16}$$

$$p_2 = \rho_1 v_1^2 \tag{17}$$

and, of course,

$$T_2 = T_1 \tag{18}$$

2.4. Energy Conservation in the Nonradiative Shock.

In the nonradiative shock, the energy loss term is zero by definition. Consider the three contributions to energy, written on a *per particle* basis. In the post-shock gas, these must add up to the total energy per particle in the pre-shock gas. For a strong shock, the thermal energy in the pre-shock gas is negligible, so the *total* energy per particle in the pre-shock gas is just $\mu v_1^2/2$.

1. Internal thermal energy. This is just $\frac{3}{2}kT$ per particle. From equation 9,

$$\frac{3}{2}kT_2 = \frac{9}{16}\frac{\mu v_1^2}{2} \tag{19}$$

2. PV energy. This is just kT per particle:

$$kT_2 = \frac{6}{16}\frac{\mu v_1^2}{2} \tag{20}$$

3. Kinetic energy. This is just

$$\frac{\mu v_2^2}{2} = \frac{1}{16}\frac{\mu v_1^2}{2} \tag{21}$$

The sum is the full $\frac{16}{16}$. Fortunately, energy is conserved! In balancing the energy, it is easy to forget the PV energy. If one uses enthalpy, the energy accounting is easier.

2.5. Structure of Radiative Shocks.

A shock is instantaneous (in the context of the present discussion). However, the energy loss term involves an integration over time, and is not instantaneous. Immediately after crossing the shock front, there has been no time for radiation and the loss term is zero. Thus, the gas *just behind the shock* has not radiated and is subject to the *nonradiative* shock relations, equations 10 - 14. As the gas moves away from the shock, the loss term becomes larger and the gas cools off. In extreme cases (which happen frequently in astronomy), it will cool to its initial temperature and the shock will, in the final analysis, be isothermal.

By comparing the solutions for nonradiative (equations 10 - 14) and isothermal (equations 15 - 18) shocks, we see that the densities and temperatures change by large factors during the cooling process. However, the pressure changes only by 25%. This is a relatively small change, and allows one to approximate the cooling as taking place at constant pressure. Thus, one can calculate the cooling using the specific heat at constant pressure c_p; on a per-particle basis, $c_p = \frac{5}{2}k$ so that one can write

$$\frac{d(\frac{5}{2}kT)}{dt} = -n\Lambda \tag{22}$$

As the gas cools under nearly constant pressure, it gets denser with $\rho \propto T^{-1}$. As the gas gets denser, the mass conservation equation (S3) means it must move more slowly. Physically, this deceleration requires a force. This force is the increased pressure (the pressure in the isothermal region is 25% *higher* than that in the nonradiative region). Everything fits together.

2.6. Velocities in the Frame of the Universe

All the velocities discussed above are with respect to the *shock*. However, we are most often concerned with velocities in some other frame. Usually we consider the ambient, unshocked, upstream gas as residing in the frame of the universe, with the shock moving through it with velocity u_{shock}; clearly, $u_{shock} = -v_1$. In this "universe" frame of the unshocked gas, the post-shock gas moves at velocity $\frac{3}{4}u_{shock}$ for a *nonradiative* shock and $(1 - \frac{c_1^2}{u_{shock}^2})u_{shock} \approx u_{shock}$ for an *isothermal* shock.

Converting among these coordinate systems requires care in adding and subtracting speeds because you must constantly keep in mind the directions in which things are moving. As the homework problems show, this is not always easy.

3. BLAST WAVES.

Blast waves are the result of the release of a large amount of energy E in a uniform medium of mass density ρ, such as happens on Earth with a nuclear bomb explosion and as happens in astronomy with a supernova in the interstellar medium. It is also possible to treat cases in which ρ depends on distance from the explosion to an arbitrary power, but we will forgo this generalization.

Blast waves create huge pressures which drive shocks into the ambient medium. In the initial stages, the radiative loss is small so the shocks can be regarded as adiabatic. In the latter stages the shocks are radiative and approximately isothermal.

3.1. Nonradiative Blast Wave: the Sedov-Taylor Case.

In the initial stages, the total energy E inside the spherical shock remains constant because there is no radiation. Furthermore, the physics of strong shocks tells us that a constant, fixed fraction f of the energy goes into kinetic energy of the post-shock gas. As measured in the frame of the shock, $f = \frac{1}{16}$.

Let M be the swept-up mass; $M = \frac{4\pi}{3}R_s^3$. In the frame of the shock, the kinetic energy is $KE = \frac{M(v_s/4)^2}{2}$; the enthalpy is 15 times larger, so $H = \frac{15M(v_s/4)^2}{2}$. In the frame of the universe, H remains unchanged but the kinetic energy is 9 times larger. In the frame of the universe, the total of H and KE must equal E; thus

$$\frac{24M(v_s/4)^2}{2} = E \tag{23}$$

We now use $M = \frac{4\pi}{3}R_s^3$ and $v_s = \frac{dR_s}{dt}$ and obtain

$$R_s^3 v_s^2 = \frac{E}{\pi\rho} \tag{24}$$

The solution to this is

$$R_s = \left(\frac{25E}{4\pi\rho}\right)^{1/5} t^{2/5} \tag{25}$$

$$v_s = \frac{2}{5}\left(\frac{25E}{4\pi\rho}\right)^{1/5} t^{-3/5} \tag{26}$$

Note that the total kinetic energy $KE \propto \rho R_s^3 v_s^2$ and is independent of time, as it must be under these conditions.

The above solutions were first obtained by the Russian and British theorists Sedov and Taylor. According to popular legend, these foreign scientists developed this theory (independently) in an afternoon and applied it to the photographs of the first nuclear test blast published in *Life* magazine in the mid 1940's, just after WWII. *Life* conveniently provided enough information (timescale, size) for them to derive E (of course, they already knew the ambient density of air ρ). Taylor sent his result to a widely-circulated British newspaper (the London *Times*?) and the American government became very upset, suspecting a major leak of information. If this story is true, it may have contributed significantly to the long-standing policies of the American government regarding military secrecy, which have been regarded by many as unnecessarily strict.

3.2. Later Radiative Stages

The radiative portion can be divided into two stages. In the first, the outer shell radiates while the hot interior, which is at a lower density and higher temperature, doesn't radiate significantly. In this case, the hot interior evolves adiabatically with $p \propto \rho^\gamma = \rho^{5/3} \propto R_s^{-5}$. The shell momentum is $P_s = Mv_s = \frac{4\pi}{3}\rho R_s^3 v_s$. The shell evolves according to Newton's law, $F = \frac{dP_s}{dt}$; the force is

$4\pi R_s^2 p \propto R^{-3}$. With this, the subsequent evolution of the shell is $R_s \propto t^{2/7}$. This is called the "pressure-dominated snowplow" phase.

Eventually even the hot interior cools off and its pressure becomes negligible. In this case the dense shell coasts through the interior, and we have the "snowplow" phase: the total momentum is constant, with $\frac{dMv_s}{dt} = 0$. This leads to $R_s \propto t^{1/4}$.

3.3. Application to Astronomy

A supernova releases about $E = 10^{51}$ erg, or $E_{51} = 1$. The average volume *number* (not mass) density of the interstellar medium is about 1 H-atom per cm^3. Writing the nonradiative equations normalized to these appropriate units with $\mu = 1.7 \times 10^{-24}$ gm, we obtain

$$R_s = 12.5 \left(\frac{E_{51}}{n}\right)^{1/5} t_4^{2/5} \text{ pc} \tag{27}$$

$$v_s = 490 \left(\frac{E_{51}}{n}\right)^{1/5} t_4^{-3/5} \text{ km/s} \tag{28}$$

$$T_s = 3.3 \times 10^6 \left(\frac{E_{51}}{n}\right)^{2/5} t_4^{-6/5} \text{ K} \tag{29}$$

Here t_4 is the time in units of 10^4 years.

The Cygnus Loop is the closest spectacular remnant. It is about 2 degrees in diameter, 4 times larger than the Sun and Moon. Its linear radius is about 18 pc, its age about 18000 yr. Its temperature is about 3×10^6 K, and it is just leaving the nonradiative phase; parts of its shell have cooled enough to emit brightly in the optical region, where the gas physics is very similar to that of HII region gas. The hot interior, at several million degrees, emits soft X-rays strongly.

The Cygnus Loop is the result of a single supernova. However, supernovae come from massive stars which have short lifetimes. Such stars tend to be borne in clusters. Suppose that N supernovae cluster in space and time; for the present purposes, we can approximate this as a single superexplosion with energy $N \times 10^{51}$ erg. Most clusters are small, but a small minority are huge; the largest contain about $N = 6000$ supernovae, but at any one time in a typical Galaxy here is only one such cluster. As a rough approximation, the total energy injected is $E_{51} = N$ and the shells expand until their velocities drop to somewhat more than the mean sound speed in the ambient interstellar medium, or to about 15 km/s. The shells produced by such clusters become very large, larger than the thickness of the gaseous disk of a spiral galaxy, and blow huge holes into the disk. These affect the physics of the interstellar medium, the gaseous galactic halo, and the overall chemical evolution of the galaxy.