RADIATIVE TRANSFER

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1. INTRODUCTION AND BASIC DEFINITIONS

We write the equation of transfer

$$\frac{dI_\nu}{ds} = \epsilon_\nu - \kappa_\nu I_\nu \tag{1}$$

where ds is positive *towards the observer*, and we normally define two quantities, the *optical depth* (opacity) τ_ν and the *source function* Σ_ν:

$$d\tau_\nu = -\kappa ds \tag{2a}$$

$$\Sigma_\nu = \frac{\epsilon_\nu}{\kappa_\nu} \tag{2b}$$

Notice that $d\tau$ is positive *away from the observer*. That is, we speak of the front surface of a cloud, or star, as as having optical depth zero, while somewhere in the deep interior of a cloud has $s = 0$. With this, the equation of transfer becomes

$$\frac{dI_\nu}{-d\tau_\nu} = \Sigma_\nu - I_\nu \tag{3}$$

For a discussion of general solutions, see Mihalis.

Consider the LTE case in which the emission process is described by a single temperature T. Then $\Sigma_\nu = B_\nu(T)$. In realistic ISM conditions, this temperature is not necessarily the kinetic temperature because collisions may not dominate the distribution. So, more generally, we define the *excitation temperature* T_x as *that temperature that gives us the proper population ratio* $\frac{n_2}{n_1}$. (Here we consider a two-level system with the upper level being 2 and the lower 1). Thus,

$$\Sigma_\nu = B_\nu(T_x) \tag{4}$$

In the case of a single two-level system, we do *not* need $T_x = T_K$; in the case of a multiple level system, such as a molecule, *each pair of levels can have a different* T_x *and, moreover, the ratios* $\frac{n_3}{n_1}$ *and* $\frac{n_3}{n_2}$ *can have different* T_x*!* So this use of T_x is completely general.

2. EXPRESSING IN TERMS OF EINSTEIN COEFFICIENTS

2.1. Some Important Relationships among Einstein Coefficients

The standard relationships among the Einstein coefficients are

$$A_{21} = \frac{2h\nu^3}{c^2} B_{21} \tag{5a}$$

$$\frac{B_{21}}{g_1} = \frac{B_{12}}{g_2} \tag{5b}$$

Now define the energy of the emitted photon in temperature units, the *transition temperature* is

$$T_{21} = \frac{h\nu}{k} \tag{6}$$

and consider an atom sitting in a blackbody radiation field whose temperature is T_{21}. The ratio of the downward radiatively induced rate to the downward spontaneous rate is

$$\frac{B_{21}J}{A_{21}} = \frac{B_{21}B_\nu(T_{21})}{A_{21}} = \frac{1}{e^1 - 1} = 0.6 \tag{7}$$

Thus we reach the important conclusion that *for a transition in a a radiation field having* $J = B_\nu(T_{21})$*, the induced rate is nearly equal to the spontaneous rate.* Of course, this conclusion is hardly new: it appears in the basic reasoning Einstein used to derive his famous coefficients.

2.2. Emission and Absorption coefficients in terms of Einstein Coefficients

For spectral lines, we can express ϵ and κ in terms of the Einstein coefficients. For a spectral line, κ_ν contains the information on line shape. Einstein coefficients give total emission/absorption integrated over the whole line, so they tell us $\kappa_\nu d\nu$. Then

$$\int \epsilon_\nu d\nu = \frac{h\nu}{4\pi} n_2 A_{21} \tag{8a}$$

expresses the rate of photon emission per steradian times the photon energy, and

$$\int \kappa_\nu d\nu = \frac{h\nu}{4\pi}(n_1 B_{12} - n_2 B_{21}) \tag{8b}$$

expresses absorptions minus stimulated emissions: κ is the *net* absorption, *accounting for stimulated emission from the upper level.* This means that κ depends on T_x.

With the standard relationships among the Einstein coefficients, and the Boltzmann distribution $\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-h\nu/kT_x}$ (Note the x in T_x !!), we have

$$\int \kappa_\nu d\nu = n_2 \frac{A_{21}c^2}{8\pi\nu^2}(e^{h\nu/kT_x} - 1) \tag{9a}$$

or, in terms of the total number of atoms (not just those in the upper state),

$$\int \kappa_\nu d\nu = n_{tot} \frac{A_{21} c^2}{8\pi\nu^2} \frac{g_2}{g_1} \frac{1 - e^{-h\nu/kT_x}}{1 + \frac{g_2}{g_1} e^{-h\nu/kT_x}} \tag{9b}$$

Combining these, we find the source function, which (as it must be) is just $B_\nu(T_x)$. :

There is some important behavior to notice in the above equation. If $T_x \rightarrow \infty$, then $\kappa \rightarrow 0$; in this limit, stimulated emissions just cancel absorptions so $\kappa \rightarrow 0$. If $T_x \rightarrow 0$, then all the atoms go to the ground state 1, so there are no stimulated emissions and

$$\int \kappa_{\nu, T_x=0} d\nu = n_{tot} \frac{A_{21} c^2}{8\pi\nu^2} \frac{g_2}{g_1} \tag{10}$$

Finally, *negative temperatures* aren't excluded: they correspond to $\frac{n_2}{n_1} > \frac{g_2}{g_1}$—the case of interstellar *masers*, with $\kappa < 0$.

Now write the source function Σ: you find that n_2, A_{21}, and ϕ_ν all cancel out so that

$$\Sigma_\nu = B_\nu(T_x) \tag{11}$$

This simply reflects the fact that, by defining $\frac{n_2}{n_1}$ in terms of a temperature, we are in effect assuming LTE; and in LTE the source function is always $\Sigma_\nu = B_\nu(T_x)$. In particular, there are no line parameters (Einstein A, shape function) in Σ!

3. THE RAYLEIGH-JEANS (RJ) LIMIT

With the equation of transfer above, we always encounter the lengthy expressions of the sort $\left[\frac{2h\nu^3}{c^2}(e^{h\nu/kT} - 1)^{-1}\right]$, which makes equations cumbersome. However, if we restrict ourselves to the RJ limit, the equations become much simpler. This is particularly so if we express the line frequency, which iis a measure of the energy difference between states 2 and 1, as temperature T_{21}:

$$\frac{h\nu}{k} \rightarrow T_{21} \tag{12a}$$

$$\frac{1}{e^{h\nu/kT} - 1} \rightarrow \frac{T}{T_{21}} \tag{12b}$$

$$B_\nu(T) \rightarrow \frac{2k\nu^2 T}{c^2} = \frac{2kT}{\lambda^2} \tag{12c}$$

With this simple way of writing things, we have for the absorption coefficient:

$$\int \kappa_\nu d\nu = n_2 \frac{A_{21}c^2}{8\pi\nu^2} \frac{T_{21}}{T_x} \tag{13a}$$

or, in terms of the total number of atoms n_{tot},

$$\int \kappa_\nu d\nu = n_{tot} \frac{A_{21}c^2}{8\pi\nu^2} \frac{T_{21}}{T_x} \frac{\frac{g_2}{g_1}}{1 + \frac{g_2}{g_1}} \tag{13b}$$

Equation 13b is very important! It shows that $\kappa \propto \frac{1}{T_x}$, so that *cold clouds have higher optical depths.* This happens simply because the upper state gets less populated at colder temperatures, reducing the ratio of stimulated emissions to absorptions.

For the source function and specific intensity, we have:

$$\Sigma_\nu = \frac{2kT_x}{\lambda^2} \tag{14a}$$

$$I_\nu = \frac{2kT_B}{\lambda^2} \tag{14b}$$

so we can write for equation 3, the fundamental equation of transfer,

$$\frac{dT_B}{-d\tau_\nu} = T_x - T_B \tag{15}$$

4. THE LINE SHAPE FUNCTION ϕ_ν

Let ϕ_ν be the *line shape function.* It is the probability per unit frequency interval that the photon is emitted; $\int \phi_\nu d\nu = 1$. As ϕ_ν gets narrower, the line-center opacity increases: $\phi_{\nu,LC} \approx \frac{1}{\delta\nu}$, where $\delta\nu$ is the line width. This means $\kappa_\nu \sim \kappa_{LC}\delta\nu\phi_\nu$, where κ_{LC} is the opacity at line center.

Lines are commonly represented by Gaussians; if thermal broadening alone determines line shape, this is exact. Sometimes it is also important to include the Lorentzian "damping wings" or pressure broadening; the combination of a Gaussian and a Lorentzian is a Voigt profile (see RL). We, however, will stick with Gaussians. For a Gaussian,

$$\phi_\nu = \frac{1}{\sqrt{\pi}\delta\nu} e^{\frac{-\Delta\nu^2}{\delta\nu^2}} \tag{16a}$$

and

$$\tau_\nu = \tau_{LC} e^{\frac{-\Delta\nu^2}{\delta\nu^2}} = \sqrt{\pi}\tau_{LC}\delta\nu\phi_\nu \tag{16b}$$

where τ_{LC} is the optical depth at line center, $\Delta\nu$ is the frequency offset from line center and $\delta\nu$ is half the full $\frac{1}{e}$ width. Observers usually use the full width at half maximum $\delta\nu_{FWHM}$, for which

$$\delta\nu_{FWHM} = 2(\ln 2)^{1/2}\delta\nu = 1.665\delta\nu \tag{17a}$$

$$\int \tau d\nu = \frac{\sqrt{\pi}}{2(\ln 2)^{1/2}}\tau_{LC}\delta\nu_{FWHM} = 1.065\tau_{LC}\delta\nu_{FWHM} \tag{17b}$$

In terms of velocity V (km s^{-1}; $\delta V = \lambda\delta\nu$),

$$\delta V_{FWHM} = 0.213\sqrt{\frac{T}{A}}\ \text{km s}^{-1} \tag{18a}$$

$$T = 21.8\ \delta V_{FWHM}^2 A \tag{18b}$$

5. SOLUTION OF RADIATIVE TRANSFER FOR A SIMPLE CASE

Suppose we know T_x as a function of z, or equivalently τ; then one can explicitly solve equation 15. In the nice case of a uniform slab in which T_x is constant, we have

$$T_B = T_x(1 - e^{-\tau_\nu}) + T_{B,BKGND}e^{-\tau_\nu} \tag{19a}$$

which has the nice simple interpretation: the first term is the emission within the slab; the second term is the emission incident from behind, attenuated by the opacity of the slab.

The line intensity is usually measured with respect to the surrounding continuum—let's call this the *line deflection* ΔT_B. If $T_{B,BKGND}$ is frequency-independent continuum, denoted by $T_{B,BC}$ (for **B**ackground **C**ontinuum), then the line deflection is

$$\Delta T_B = T_B - T_{B,BC} = (T_x - T_{B,BC})(1 - e^{-\tau_\nu}) \tag{19b}$$

Note that we have either an *emission* or *absorption* line, depending on the sign of $(T_x - T_{B,BC})$. In other words, clouds that are colder than the background produce absorption lines.

5.1. The 21-cm Line

To a good approximatioo, The 21-cm line has $T_x = T_K$ because the critical density n_{crit} is small. Moreover, because of the low frequency ($T_{21} = 0.068$ K) the RJ approximation applies. We have the interesting limits, first for the combination ($\tau_{LC} \ll 1$) and ($T_{B,BC} \ll T_K$):

$$\int \Delta T_B dV \rightarrow \frac{N(HI)}{1.83 \times 10^{18}} \tag{20}$$

which means that the integrated line intensity $\propto$ the HI column density and is independent of T_K. This, plus the fortunate circumstances that the 21-cm line is, in fact, usually fairly optically thin and $T_{B,BC}$ is small, are of crucial importance for 21-cm line surveys: they provide the total HI column density.

The other interesting limit is, of course, $\tau_{LC} \gg 1$:

$$T_{B,LC} \rightarrow T_K \tag{21}$$

so it's equivalent to being inside a blackbody at temperature T_K—as it must be. Note that $T_{B,LC} - T_{B,BC} = T_K - T_{B,BC}$: the line can be in absorption or emission, but in both cases case $T_{B,LC} \rightarrow T_K$, independent of $T_{B,BC}$—which, of course, makes sense because the background continuum is completely absorbed.