Lab #4: Astrometric Orbit Determination

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Your report is due on November 17, 2009 at 6:00 PM PDT.

1 Overview

1.1 Schedule

This is a three-week lab, starting 10/27, with show-and-tell on 11/3 and 11/10. Your lab report is due on 11/17. For show-and-tell on 11/3 you should have progressed at least to step 6 (see below) and for 11/10 you should have reached step 8.

This is a sophisticated lab that builds on previous work and, unlike previous labs, develops a detailed mathematics framework within which to interpret your data. One of the main lessons from this lab is that simple physical problems (the two-body problem in Newtonian gravity) need complex analysis. Be sure to pace yourself!

1.2 Goal

- Measure the position of an asteroid at multiple epochs and estimate the Keplerian orbital elements.

1.3 Key steps

Execute the following steps for this lab:

1. Work though the details of orbital motion (§2) and understand the definitions of the Keplerian orbital elements and ecliptic and equatorial coordinates. As an example, take the orbital elements for Ceres in Table 4 and compute and plot the orbital separation, r, the true anomaly, v, and the x- and y-coordinates in the orbital plane as a function of time. You should be able to reproduce the plots seen in Figure 2, Figure 3, and Figure 4
2. Measure and report the position of an asteroid in Kitt Peak Super-LOTIS images at four epochs. The measurements must span several days to give reliable results.
3. For the first three epochs, convert the measured coordinates from equatorial to Cartesian ecliptic coordinates and find the components of the target unit vector $\mathbf{s}$ (see §4).
4. Compute the first and second time derivatives of $\mathbf{s}$ using a Taylor series approximation (see §3).
5. Become familiar with the web-based JPL HORIZONS[1] ephemeris and use it to compute the heliocentric Cartesian ecliptic coordinates of the earth on your observation dates.
6. Compute r, ρ, and $d\rho/dt$ for your asteroid.
7. Compute $\mathbf{r}$ and $d\mathbf{r}/dt$.
8. Derive the Keplerian orbital elements from $\mathbf{r}$ and $d\mathbf{r}/dt$.
9. Use your estimated orbital elements to predict the geocentric equatorial coordinates of the asteroid at the fourth epoch and compare the measured and predicted positions of the asteroid. Discuss the errors in the orbital elements and your predicted position.

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[1] http://ssd.jpl.nasa.gov/horizons.cgi

Notice that we have not used the method least squares to compute the orbital elements. If you have executed steps 1–9 and have written up your report consider why you cannot use the method of linear least squares for this problem. Ask one of your instructors how to incorporate your code that computes geocentric equatorial coordinates into the non-linear least-squares IDL program `MPFIT`. Use the Keplerian orbital elements from Laplace's method (step 8) as your initial guess and use data from four or more epochs to find the best fit orbital elements.

2 Orbital motion

Consider the orbital motion of two masses m_1 and m_2, located at $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to the center of mass of the system. Conservation of momentum requires that $\mathbf{r}_1 = -m_2\mathbf{r}/(m_1 + m_2)$ and $\mathbf{r}_2 = m_1\mathbf{r}/(m_1 + m_2)$, where $\mathbf{r} = \mathbf{r}_2$-$\mathbf{r}_1$ is the vector jointing the two objects. The relative motion is described by

$$\ddot{\mathbf{r}} = -k^2 \frac{\mathbf{r}}{r^3}, \qquad (1)$$

where dots denote time derivatives, the constant $k^2 = G(m_1+m_2)$, and G is Newton's constant. In this application we consider the case where m_1 is the mass of the sun and $m_1 >> m_2$. For convenience we use units where mass, length, and time are measured in solar masses, AU, and days, respectively and $k = \sqrt{GM_\odot} = 0.017\,202\,098\,950$ AU$^{3/2}$ d^{-1}. In this system the period (in days) is given by Kepler's third law stated as $p = 2\pi\, a^{3/2}/k$, where a is the semimajor axis (in AU).

The cross product of Eq. (1) with $\mathbf{r}$ yields

$$\mathbf{r} \times \ddot{\mathbf{r}} = 0, \qquad (2)$$

because $\mathbf{r} \times \mathbf{r} = 0$. On integrating with respect to time we find

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}, \qquad (3)$$

where the constant of integration, **h,** is the specific[2] angular momentum. In the natural coordinate system of the orbit the z-axis points in the same direction as $\mathbf{h}$ because the triple product

$$\mathbf{r} \cdot \mathbf{h} = \mathbf{r} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \dot{\mathbf{r}} \cdot (\mathbf{r} \times \mathbf{r}) = 0,$$

and $\mathbf{r}$, which defines the orbit, lies in the x-y plane.

The dot product of Eq. (1) with $\dot{\mathbf{r}}$ gives

$$\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -\frac{k^2}{r^3}\dot{\mathbf{r}} \cdot \mathbf{r}. \qquad (4)$$

[2] In this context specific means per unit mass.

The product $\mathbf{r}\cdot\dot{\mathbf{r}}$ is the component of velocity in the radial direction times r, thus $\mathbf{r}\cdot\dot{\mathbf{r}} = r\dot{r}$ and Eq. (4) yields

$$\int \dot{\mathbf{r}} \cdot \frac{d\dot{\mathbf{r}}}{dt} dt = -k^2 \int \frac{1}{r^2} \frac{dr}{dt} dt \ , \tag{5}$$

or

$$\int \dot{\mathbf{r}}\, d\dot{\mathbf{r}} = -k^2 \int dr/r^2 \ , \tag{6}$$

which, integrates to

$$\frac{1}{2}|\dot{\mathbf{r}}|^2 - \frac{k^2}{r} = \mathcal{E} \ , \tag{7}$$

where $|\dot{\mathbf{r}}|^2 = V^2$. We recognize Eq. (7) as conservation of energy, where the two terms on the left hand side correspond to the specific kinetic and gravitational potential energy, respectively.

Conservation of angular momentum (Eq. (3)) and energy (Eq. (7)) can be expressed in scalar form if we adopt cylindrical polar coordinates with the z-axis in the direction of $\mathbf{h}$. As $\mathbf{r}\cdot\mathbf{h} = 0$, the orbit lies in the x-y plane and the Cartesian componemts of $\mathbf{r}$ and $\mathbf{h}$ are

$$\begin{aligned} \mathbf{r} &= r(\cos\theta, \sin\theta, 0) \\ \mathbf{h} &= h(0,0,1). \end{aligned} \tag{8}$$

In this coordinate system Eq. (3) yields the scalar relation

$$r^2\dot{\theta} = h \ , \tag{9}$$

and conservation of energy, Eq. (7), using

$$V^2 = \dot{r}^2 + (r\dot{\theta})^2 \tag{10}$$

can be written as

$$\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{k^2}{r} = +\mathcal{E} \ . \tag{11}$$

Eliminating $\dot{\theta}$ from Eq. (11) using Eq. (9) yields

$$\frac{1}{2}V^2 = \frac{1}{2}\left(\dot{r}^2 + \frac{h^2}{r^2}\right) = \frac{k^2}{r} + \mathcal{E} \ . \tag{12}$$

2.1 Bound and unbound orbits

We can define the quantity a such that constant

$$\mathcal{E} = -\frac{k^2}{2a}. \tag{13}$$

In Eq. (7), when the total energy is zero the separation of the two bodies approaches infinity as V tends to zero, and the orbit is parabolic. When $\mathcal{E} < 0$ the orbit is elliptical and the quantity a, known as the semimajor axis, is positive. Using Eq. (12) and the definition Eq. (13) we can write the magnitude of the total velocity as

$$V^2 = \dot{r}^2 + \frac{h^2}{r^2} = k^2\left(\frac{2}{r} - \frac{1}{a}\right). \tag{14}$$

As $V^2 > 0$ we have that $r < 2a$, and the orbit is bound.

Table 1: Orbital quantities and their units. Although the units of angles must be in radians for numerical computations, they are often listed in degrees. The six Keplerian orbital elements are indicated. Other quantities are either derived or a function of time.

Name	Symbol	Orbital element
Semimajor axis [AU]	a	✓
Epoch of perihelion [Julian date]	τ	✓
Current epoch [Julian date]	t	
True anomaly [rad]	v	
Argument of perihelion [rad]	ω	✓
Polar angle from the x-axis [rad]	$\theta = v + \omega$	
Longitude of ascending node [rad]	Ω	✓
Inclination [rad]	i	✓
Eccentricity	e	✓
Eccentric anomaly [rad]	E	
Mean motion [rad/day]	$n = ka^{-3/2}$	
Mean anomaly [rad]	$M = n(t - \tau)$	
Orbital period [days]	$p = 2\pi/n$	

2.2 Finding the position

A significant point in the object's orbit is perihelion (the closest approach to the sun). The time of perihelion is denoted as τ (see in Figure 1). At some later time, t, the body has moved through an angle v, which is called the true anomaly. The polar angle measured from the x-axis is

$$\theta = v + \omega \tag{15}$$

where ω is the argument of perihelion.

Our task is now to find the body's position given by the separation and angle (r, v) as a function of time. We integrate Eq. (14) by first writing it as

$$\dot{r}^2 = -\frac{k^2}{ar^2} F(r), \tag{16}$$

where the function F is defined as

$$F(r) = r^2 - 2ar + h^2 a / k^2 \ . \tag{17}$$

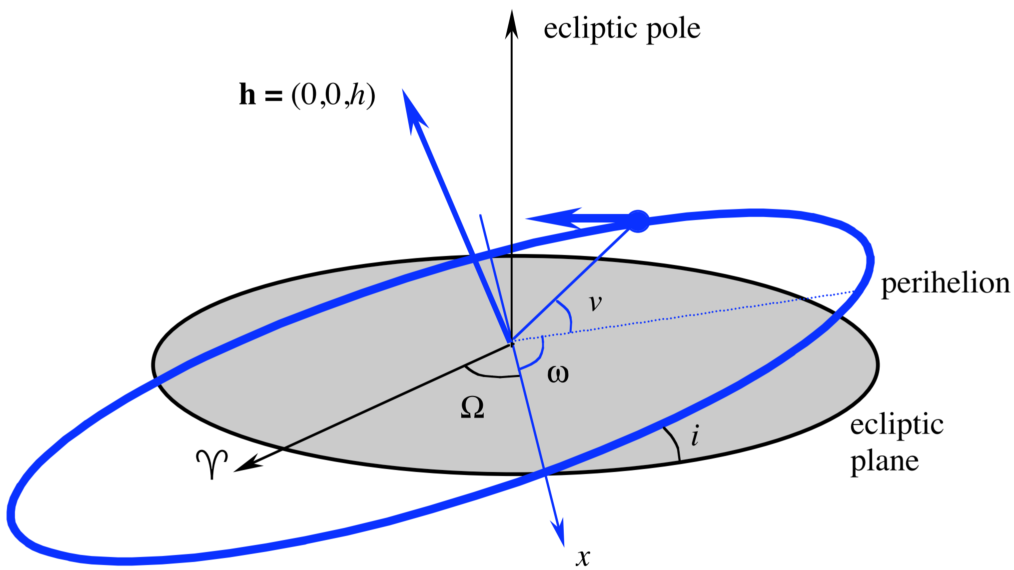

Figure 1: The orbit in the ecliptic coordinate frame. The angular momentum vector h establishes the orientation of the orbit and determines the angles Ω (longitude of the ascending node) and the orbital inclination, *i.* The angles ω and *v* are the argument of perihelion and the true anomaly, respectively. The ecliptic coordinate system is defined by the direction ♈, the northern vernal equinox at which point the sun crosses the celestial equator from south to north and the celestial pole, which is normal to the orbital plane of the earth—the ecliptic plane.

As $\dot{r}^2$, k^2, r^2, and a are all positive, the quadratic function F must be negative—the extreme values of r are found by setting $F(r) = 0$ with roots r_1, and r_2, which are the perihelion and aphelion distances. If we define

$$e = \frac{r_2 - r_1}{r_2 + r_1} \tag{18}$$

then

$$r_1 = a(1-e); \quad r_2 = a(1+e), \tag{19}$$

and $r_1 + r_2 = 2a$.

Conventionally, the radius is written in parametric form as

$$r = a(1 - e\cos E), \tag{20}$$

where the angle E is known as the eccentric anomaly, which is zero at perihelion and increases by 2π every orbit. This change of variable allows us to write F as

$$F(r) = -(ae\sin E)^2 . \tag{21}$$

and thereby integrate Eq. (16) using the change of variable to E as

$$\int_0^E (1 - e\cos E')dE' = ka^{-3/2}\int_\tau^t dt', \tag{22}$$

as $E = 0$ at perihelion Eq. (22) yields Kepler's equation

$$M = n(t-\tau) = E - e\sin E, \tag{23}$$

where we have defined the mean motion n, so that $n^2a^3 = k^2$ (Kepler's third law), and introduced the angle known as the mean anomaly, M, such that

$$M = n(t-\tau). \tag{24}$$

The mean anomaly is useful because it is linearly proportional to time, unlike the angles v or E. The units of n are radians per day; thus, the orbital period in days is $p = 2\pi/n$.

2.3 Integrating Kepler's equation

The eccentric anomaly, E, is useful because the radius vector, r, and the time, t, have been written in terms of it via Eqs. (20) and (23). Furthermore, we can also relate E to the true anomaly, v. From Eq. (17) we have

$$r_1 r_2 = a\frac{h^2}{k^2}, \tag{25}$$

which together with Eq. (19) implies

$$h^2 = k^2 a(1-e^2). \tag{26}$$

Furthermore, combining Eqs. (23) and (26) we have

$$\frac{dv}{dE} = \frac{\sqrt{1-e^2}}{1-e\cos E} \,. \tag{27}$$

By changing the variable to $\tan E/2$ the integral is

$$\int_0^v dv' = \int_0^E \frac{\sqrt{1-e^2}}{1-e\cos E'} dE' = 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan E/2\right], \tag{28}$$

and the true anomaly is

$$v = 2\arctan\left[\sqrt{\frac{1+e}{1-e}}\tan E/2\right]. \tag{29}$$

We now can compute the position in terms of (r, v) using the eccentric anomaly (Eqs. (20) & (29)), so we have a complete solution. The only remaining problem is to solve Kepler's equation to compute the angle E at any time t.

2.4 Numerical solution of Kepler's equation

Figure 2 plots the mean anomaly, M, and the eccentric anomaly, E, for the asteroid Ceres using the orbital elements listed in Table 4. Computation of M using Eq. (24) is simple. However, Kepler's equation for E is not an explicit relation, and the equation must be solved numerically.

For a circular orbit, $e = 0$, it is evident that $E = M$, and this is a good guess even for non-circular orbits. If we use $E_0 = M$ as a first approximation to E, we need to find a way to improve our guess. The angle M can be considered a function of E by writing Kepler's equation, Eq. (23), as

$$M(E) = E - e\sin E \,. \tag{30}$$

The Taylor series expansion of Eq. (30) is

$$\begin{aligned} M(E_0 + \delta E) &= M(E_0) + \left.\frac{\partial M}{\partial E}\right|_{E_0} \delta E + \cdots \\ &= M(E_0) + (1 - e\cos E_0)\delta E + \cdots \end{aligned} \tag{31}$$

suggesting that a better approximation for the eccentric anomaly is $E_1 = E_0 + \delta E$, where

$$\delta E = \frac{M - M(E_0)}{1 - e\cos E_0} \,, \tag{32}$$

or more generally,

$$E_{n+1} = E_n + \frac{M - M(E_n)}{1 - e\cos E_n}. \tag{33}$$

Successive application of Eq. (33) yields an improved estimate for E. For the example in Figure 2, starting with $E_0 = M$, only four iterations of Eq. (33) are needed to find E to better than 1 nrad (0.2 milli arc seconds).

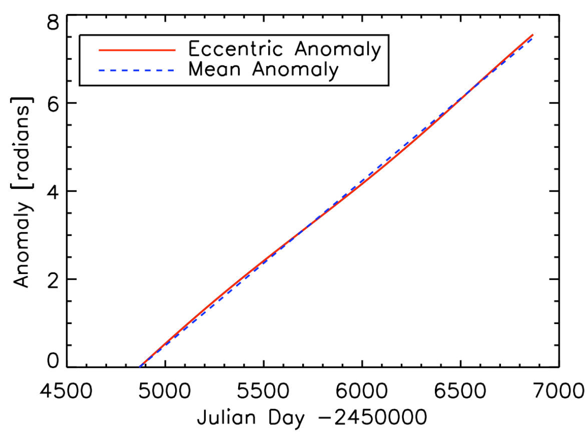

Figure 2: The mean anomaly, *M*, and eccentric anomaly, *E*, for the asteroid Ceres using the orbital elements tabulated in Table 4. The eccentricity of Ceres is small (e=0.079) and the difference between these two angles is small.

Once we have the eccentric anomaly we can find the orbital separation, r, and the true anomaly, v, from Eq. (20) and Eq. (29), respectively. The results for Ceres are shown in Figure 3.

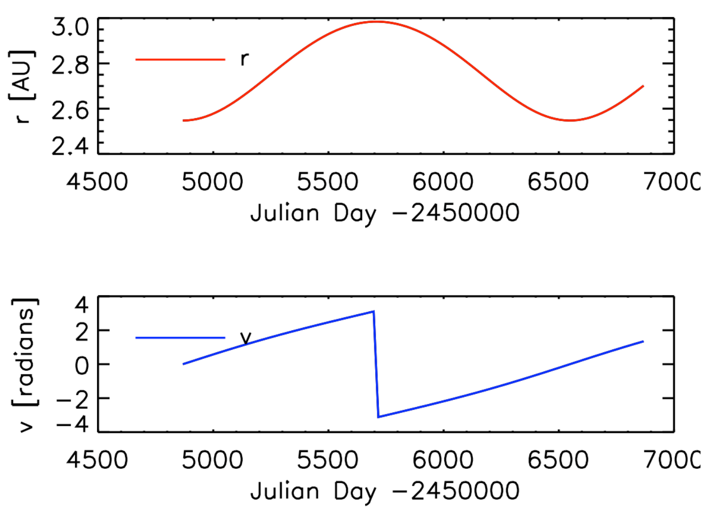

Figure 3: The orbital separation, *r*, and true anomaly, *v*, as a function of time for Ceres.

In our natural coordinate system, the z-axis is parallel to **h** and the radius vector lies in the x-y plane. The polar angle $\theta = \nu + \omega$ is measured from the x-axis and the Cartesian components of **r** are

$$\mathbf{r} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{pmatrix}. \tag{34}$$

Using these conventions, the orbital path of Ceres is plotted in Figure 4.

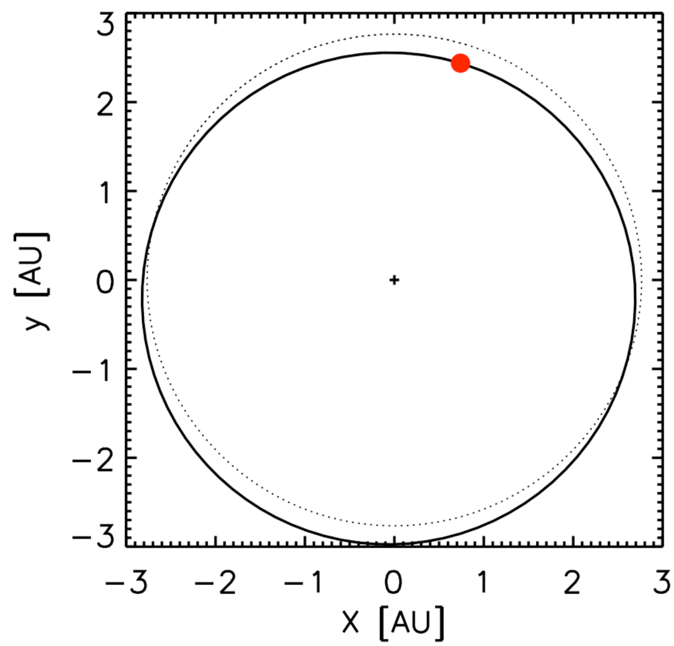

Figure 4: The position of Ceres in the plane of the orbit (the *z*-direction is parallel to h). The red dot shows the location of perihelion. The dotted line is a circle with radius equal to the semimajor axis. The orbital period of Ceres is $2\pi\, a^{3/2}/k$= 1680.3 days.

2.5 Ecliptic & equatorial coordinates

In the previous section we computed the x- and y-components of **r** in the plane perpendicular to the orbital angular momentum **h**. By convention, orbital elements for objects in the solar system are referenced to the coordinate system defined by the orbit of the Earth around the Sun. The ecliptic z-axis points towards the ecliptic pole, which is perpendicular to the earth's orbital plane, and the ecliptic x-axis is points towards the direction of the vernal equinox as shown in Figure 1. Thus, there are two rotations from the natural coordinate system of the asteroid orbit to the ecliptic system.

The first transformation is effected by a rotation about the x-axis by an angle equal to the inclination of the orbit relative to the ecliptic plane that is described by the rotation matrix

$$\mathbf{T}_x(-i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix}, \tag{35}$$

followed by a rotation about the z-axis is by an angle equal to the longitude of the ascending node that is described by

$$\mathbf{T}_z(-\Omega) = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{36}$$

The combined rotation is

$$\mathbf{T}_z\mathbf{T}_x = \begin{pmatrix} \cos\Omega & -\sin\Omega\cos i & \sin\Omega\sin i \\ \sin\Omega & \cos\Omega\cos i & -\cos\Omega\sin i \\ 0 & \sin i & \cos i \end{pmatrix}. \tag{37}$$

Thus, the Cartesian components of $\mathbf{r}$ in the ecliptic frame are given by $\mathbf{T}_z\mathbf{T}_x\mathbf{r}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = r\begin{pmatrix} \cos\Omega\cos\theta - \sin\Omega\cos i\sin\theta \\ \sin\Omega\cos\theta + \cos\Omega\cos i\sin\theta \\ \sin i\sin\theta \end{pmatrix} \tag{38}$$

where the angle $\theta = \nu + \omega$, see Eq. (15). Appendix 4 explains how to convert from ecliptic (x, y, z) to equatorial (x_{eq}, y_{eq}, z_{eq}) using Eq. (63).

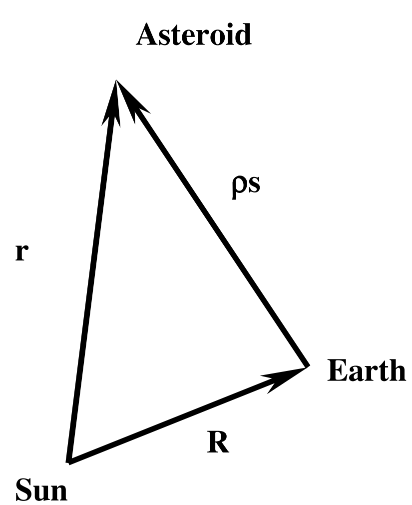

Figure 5: The relative position of the sun, earth (observer), and the target asteroid.

This transformation gives us the position in heliocentric coordinates—we have not applied the correction required to give the geocentric coordinates. Let **R** = (X_{eq}, Y_{eq}, Z_{eq}) be the position[3] vector of the Earth relative to the Sun (see Figure 5) and let the geocentric position vector of the asteroid be ρ**s**, then

$$\mathbf{r} = \mathbf{R} + \rho\mathbf{s}\,, \tag{39}$$

where **s** is the unit vector from the earth towards the asteroid. From the components of Eq. (39) in the equatorial frame we have

$$\rho\mathbf{s} = \begin{pmatrix} x_{eq} - X_{eq} \\ y_{eq} - Y_{eq} \\ z_{eq} - Z_{eq} \end{pmatrix} = \rho \begin{pmatrix} \cos\alpha\cos\delta \\ \sin\alpha\cos\delta \\ \sin\delta \end{pmatrix} \tag{40}$$

(compare with Eq. (59)). Thus, the geocentric equatorial coordinates are found using inverse trig functions from

$$\tan\alpha = \frac{y_{eq} - Y_{eq}}{x_{eq} - X_{eq}} \tag{41}$$

and

$$\sin\delta = \frac{z_{eq} - Z_{eq}}{\rho} \tag{42}$$

where

$$\rho^2 = \left(x_{eq} - X_{eq}\right)^2 + \left(y_{eq} - Y_{eq}\right)^2 + \left(z_{eq} - Z_{eq}\right)^2 . \tag{43}$$

In applications like Eq. (41) use the two-argument IDL arctan function `ATAN(Y,X)`. Be aware that the `ATAN(Y,X)` returns an angle in the range –π to π, whereas the right ascension, by convention, is in the range 0 to 2π.

3 Laplace's method for orbit determination

The orbit of solar system bodies can be determined from measurement of the object's position on the celestial sphere, Newtonian dynamics, and knowledge of the earth's orbit about the sun. If **R** and **r** are the position vectors of the Earth and the target body relative to the sun (see Figure 5), then equations of motion are (see Eq. (1))

$$\ddot{\mathbf{r}} = -k^2 \frac{\mathbf{r}}{r^3}$$

[3] The JPL HORIZONS ephemeris lists the Cartesian components of the sun-earth vector **R** in either ecliptic or equatorial coordinates. Be sure to choose the right system or use Eqs. (61) or (63) transform to the correct frame.

and

$$\ddot{\mathbf{R}} = -k^2 \frac{\mathbf{R}}{R^3}.$$

Our equations neglect the mass of the smaller body in both cases. We assume that the earth-sun vector, **R**, is known; our data are the measurements of **s**, the unit vector from the Earth to the target body.

Differentiating Eq. (39) with respect to time gives the orbital velocity with respect to the sun

$$\dot{\mathbf{r}} = \dot{\mathbf{R}} + \rho\dot{\mathbf{s}} + \dot{\rho}\mathbf{s}\,, \tag{44}$$

and differentiating again gives the acceleration

$$\ddot{\mathbf{r}} = \ddot{\mathbf{R}} + \ddot{\rho}\mathbf{s} + 2\dot{\rho}\dot{\mathbf{s}} + \rho\ddot{\mathbf{s}}.$$

Substituting into the equations of motion and eliminating **r** leads to

$$\mathbf{s}\left(\ddot{\rho} + k^2 \frac{\rho}{r^3}\right) + 2\dot{\rho}\dot{\mathbf{s}} + \rho\ddot{\mathbf{s}} = k^2\mathbf{R}\left(\frac{1}{R^3} - \frac{1}{r^3}\right).$$

To eliminate the first term take the cross product with **s** and then the dot product[4] with $d\mathbf{s}/dt$

$$\rho = k^2\left(\frac{1}{R^3} - \frac{1}{r^3}\right)\frac{\dot{\mathbf{s}}\cdot(\mathbf{R}\times\mathbf{s})}{\dot{\mathbf{s}}\cdot(\ddot{\mathbf{s}}\times\mathbf{s})}. \tag{45}$$

Together with

$$r^2 = \rho^2 + R^2 + 2\rho\mathbf{R}\cdot\mathbf{s}\,, \tag{46}$$

which is derived by squaring Eq. (39) we can solve iteratively for ρ, given an initial guess for r.

If instead of the dot product with $d\mathbf{s}/dt$ we take the dot product with $d^2\mathbf{s}/dt^2$ we find

$$\dot{\rho} = \frac{k^2}{2}\left(\frac{1}{R^3} - \frac{1}{r^3}\right)\frac{\ddot{\mathbf{s}}\cdot(\mathbf{R}\times\mathbf{s})}{\ddot{\mathbf{s}}\cdot(\dot{\mathbf{s}}\times\mathbf{s})}. \tag{47}$$

As the position and velocity vector at a given instant are now known the orbital elements can be computed (see § 3.2.1). Note, that to get the orbital velocity of the asteroid relative to the sun we need to employ Eq. (44).

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[4] Vector triple products are easily computed in IDL using the `DETERM` function.

3.1 Finding unit vectors and their time derivatives

To find the orbital elements we need to measure **s**, its first and second time derivates. To measure a position needs a measurement at one epoch, to measure a velocity requires two epochs, and to measure an acceleration requires three measurements, say at t_1, t_2, and t_3. The velocity and acceleration are expressed using Taylor series approximations for the observed unit vectors $\mathbf{s}_1$, $\mathbf{s}_2$, and $\mathbf{s}_3$ and solving the resultant simultaneous equations for the time derivatives. Thus, if $\tau_1 = t_2 - t_1$ and $\tau_3 = t_3 - t_2$ then

$$\begin{aligned} \mathbf{s}_1 &= \mathbf{s}_2 - \tau_1 \dot{\mathbf{s}}_2 + \frac{1}{2}\tau_1^2 \ddot{\mathbf{s}}_2 \\ \mathbf{s}_3 &= \mathbf{s}_2 + \tau_3 \dot{\mathbf{s}}_2 + \frac{1}{2}\tau_3^2 \ddot{\mathbf{s}}_2 \end{aligned},$$

yielding

$$\begin{aligned} \dot{\mathbf{s}}_2 &= \frac{\tau_3(\mathbf{s}_2 - \mathbf{s}_1)}{\tau_1(\tau_1 + \tau_3)} + \frac{\tau_1(\mathbf{s}_3 - \mathbf{s}_2)}{\tau_3(\tau_1 + \tau_3)} \\ \ddot{\mathbf{s}}_2 &= \frac{2(\mathbf{s}_3 - \mathbf{s}_2)}{\tau_3(\tau_1 + \tau_3)} - \frac{2(\mathbf{s}_2 - \mathbf{s}_1)}{\tau_1(\tau_1 + \tau_3)} \end{aligned}. \qquad (48)$$

3.2 Laplace's method applied to Ceres

Consider the position of the minor planet Ceres on three subsequent days in 2008 August shown in Table 2. The corresponding target unit vector **s**, and the time derivatives are listed in Table 3. Notice that Table 2 lists the position of Ceres in ecliptic coordinates not equatorial coordinates because ecliptic coordinates define the conventional coordinate system for computing the orbital elements. Computation of the unit vector **s** from position in ecliptic coordinates (λ, β) is afforded using Eq. (57).

Table 2: Geocentric ecliptic longitude & latitude for Ceres. The Cartesian components of the Sun-Earth vector are from the JPL HORIZONS ephemeris in ecliptic coordinates. Note that both R and *d*R/*dt* are needed for Laplace's method, and both are available from the ephemeris—only R is listed here.

UT Date 2008/08	Julian day[5]	Ecliptic longitude & latitude for Ceres [deg.]		Cartesian sun-earth vector components of **R** in ecliptic coordinates [AU]		
		λ	β	*X*	*Y*	*Z*
24.0	2454702.5	121.7592648	4.0625653	0.8849686471	-0.4888489729	4.466373306E-06
25.0	2454703.5	122.1865441	4.0992581	0.8928865393	-0.4737871683	4.402701086E-06
26.0	2454704.5	122.6133849	4.1361592	0.9005490495	-0.4585878955	4.483801584E-06

[5] Computed using the IDL function JULDAY.

Table 3: Derived geocentric target unit vector & time derivatives. The Cartesian components are given in the ecliptic coordinate system using Eq. (57) & (58).

Vector	*x*	*y*	*Z*
S	-0.53131489	0.84415310	0.071484533
d**s**/dt [day^{-1}]	-0.0062674833	-0.0039990028	0.00064058483
d^2**s**/dt^2 [day^{-2}]	3.6914851e-05	-4.3035117e-05	3.5967350e-06

Using these data and the method outlined above yields $\rho = 3.448$ AU and $r = 2.623$ AU at the midpoint. The JPL HORIZONS ephemeris gives $\rho = 3.419$ AU and $r = 2.596$ AU for UT 2008 8 25.0, so the error introduced using the Taylor series expansion for velocity and acceleration in this case is about 1%.

3.2.1 Orbital elements from Laplace's method

At the midpoint of our three asteroid observations we know the magnitude of the radius vector, r, and the total velocity, V. The semimajor axis is found by rearranging the velocity formula, Eq. (14) to give

$$a = \frac{k^2 r}{2k^2 - rV^2} \,. \tag{49}$$

Using Eq. (3) we can compute the specific angular momentum **h**, as we know both $\mathbf{r}$ and $\dot{\mathbf{r}}$ (see Eq. (3)). The vector **h** determines the orientation of the orbit in space. So far we have only considered the natural coordinate system for the orbit, where the z-axis and **h** are aligned.

We saw in § 2.5 there are two rotations from the natural coordinate system of the asteroid orbit where

$$\mathbf{h} = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \tag{50}$$

to the ecliptic system of coordinates. But, if we measure $\mathbf{r}$ and $\dot{\mathbf{r}}$ in the ecliptic frame, then as in § 2.5 the Cartesian components of **h** are also in the ecliptic frame and are given by $\mathbf{T}_z(-i)\mathbf{T}_z(-\Omega)\mathbf{h}$. Thus,

$$\begin{aligned} h_x &= h \sin\Omega \sin i \\ h_y &= -h \cos\Omega \sin i \,. \\ h_z &= h \cos i \end{aligned} \tag{51}$$

By inspection of Eq. (51) we can find the orbital elements Ω and i because

$$\tan\Omega = -h_x / h_y \tag{52}$$

and

$$\cos i = h_z / h \, . \tag{53}$$

Eq. (52) is another example where we should be careful about sign of the angle and use the two argument version of the IDL arctan function `ATAN(Y,X)`.

To find the eccentricity we rearrange Eq. (26), which gives

$$e = \sqrt{1 - \left(h^2 / ak^2\right)} \, . \tag{54}$$

Before we find the last two Keplerian orbital elements—the argument of perihelion, ω, and the epoch, τ—we need to compute the anomalies. The eccentric anomaly can be found by rewriting Eq. (20) as

$$\cos E = \frac{a - r}{ae} \, . \tag{55}$$

The sign of E depends on the body's radial velocity: $E > 0$ when the radial velocity is positive (between perihelion and aphelion) and $E < 0$ when the radial velocity is negative (from aphelion to perihelion). The true anomaly, v, follows directly from Eq. (29) and the mean anomaly from Kepler's equation, Eq. (23). From Figure 1 we have

$$\cos(v + \omega) = \frac{x \cos\Omega + y \sin\Omega}{r} \, . \tag{56}$$

Again some care is required when computing the sign of $v + \omega$—the angle is positive when $z > 0$. The final step is to find the time or perihelion, τ, from Eq. (24).

The results for the Ceres example are listed in Table 4.

Table 4: Comparison of the true and estimates orbital elements for Ceres.

	a	Ω	i	e	ω	τ
	[AU]	[deg]	[deg]		[deg]	[Julian Day]
True	2.766	80.72	10.61	0.079	73.12	2454868
Estimated	2.947	80.65	10.56	0.125	63.20	2454833
Error	-0.18	0.07	0.05	-0.05	9.9	35

4 Appendix: Ecliptic & equatorial coordinates

The most convenient frame of reference for describing orbital motion is the ecliptic frame. The x-axis direction corresponds to the line defined by the intersection of the celestial equator and the orbital plane of the earth (the ecliptic plane). The perpendicular to the ecliptic plane defines the z-axis. The y-axis forms a right-handed set with x and z thus defined. The positive x direction is defined by the earth-sun direction when the sun appears to cross the celestial equator at the Vernal equinox.

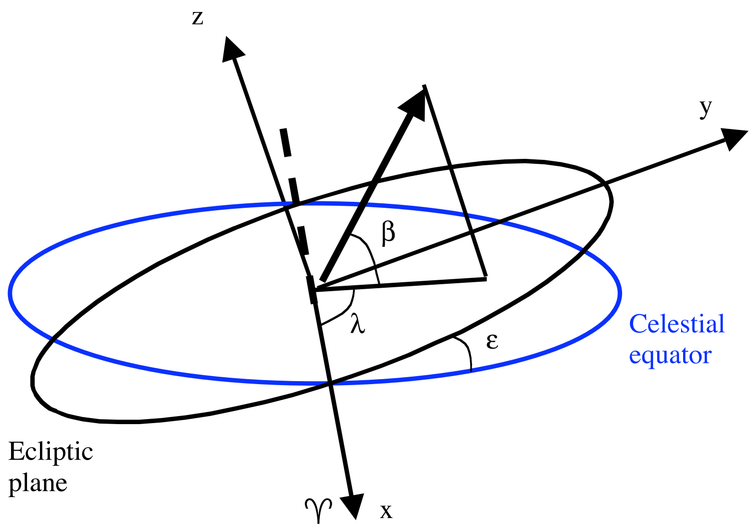

Figure 6: Ecliptic coordinates. The ecliptic represents the orbital plane of the earth about the sun, and the equator is the celestial equator. The angles λ and β are ecliptic longitude and latitude, respectively. The x-axis points towards ♈, the vernal equinox, and the *z*-axis is the ecliptic pole. The obliquity of the ecliptic or the angle between the celestial equator and the ecliptic plane is ε = +23°.43929111 for equinox J2000.

From inspection of Figure 6, the conversion between polar ecliptic and Cartesian ecliptic coordinates is given by

$$\begin{aligned} x &= \cos\lambda\cos\beta \\ y &= \sin\lambda\cos\beta \ . \\ z &= \sin\beta \end{aligned} \tag{57}$$

Note that the components of the unit vector **s** in §3 are

$$\mathbf{s} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{58}$$

The coordinates of astronomical objects are typically measured by the angles known as right ascension, α, and declination, δ, in the equatorial system defined by reference stars measured in the International Celestial Reference System, taken at epoch 2000, so the Cartesian equatorial coordinates are

$$\begin{aligned} x_{eq} &= \cos\alpha\cos\delta \\ y_{eq} &= \sin\alpha\cos\delta \ . \\ z_{eq} &= \sin\delta \end{aligned} \tag{59}$$

The ecliptic and equatorial systems are related by a rotation, ε, about the x-axis, so a rotation matrix gives the transformation from equatorial to ecliptic coordinates

$$\mathbf{x} = \mathbf{T}_x(\varepsilon)\mathbf{x}_{eq} \tag{60}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & \sin\varepsilon \\ 0 & -\sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} x_{eq} \\ y_{eq} \\ z_{eq} \end{pmatrix}, \tag{61}$$

where $\varepsilon = 23°.43929111$ for the equinox 2000.

When you apply Laplace's method to the Super-LOTIS observations your measurements will be in celestial coordinates—(α, δ). However, you do not need to convert from celestial coordinates to ecliptic coordinates, (λ, β) because all you need are the components of **s** in the ecliptic frame, i.e., you first compute (x_{eq}, y_{eq}, z_{eq}) from (α, δ) using Eq. (59), and then use Eq. (61) to find the components of **s** in the ecliptic Cartesian frame—(x, y, z).

For non-linear least squares fitting you will, however, have to compute the equatorial coordinates. Recall that the rotation matrices are orthogonal matrices, i.e., $\mathbf{A}^{\mathrm{T}}\mathbf{A}=1$, and the coordinate transformation from ecliptic coordinates to equatorial coordinates is

$$\mathbf{T}_x^T(\varepsilon)\mathbf{x} = \mathbf{T}_x^T(\varepsilon)\mathbf{T}_x(\varepsilon)\mathbf{x}_{eq} = \mathbf{x}_{eq}, \tag{62}$$

so that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_{eq} \\ y_{eq} \\ z_{eq} \end{pmatrix} \tag{63}$$

and we can find (α, δ) from Eq. (59)

$$\begin{aligned} \tan\alpha &= y_{eq}/x_{eq} \\ \sin\delta &= z_{eq} \end{aligned}. \tag{64}$$

Remember that α is defined in the range 0–2π, but inverse trig functions typically give the angle in the range $-\pi$ to π.

5 Computing positions: 10 Hygeia

Orbital elements for 10 Hygeia are listed in Table 5. The equatorial coordinates for the asteroid for an interval of 15 years are shown in Figure 7. The orbital period of Hygiea is 5.56 years, so this plot spans approximately three orbits.

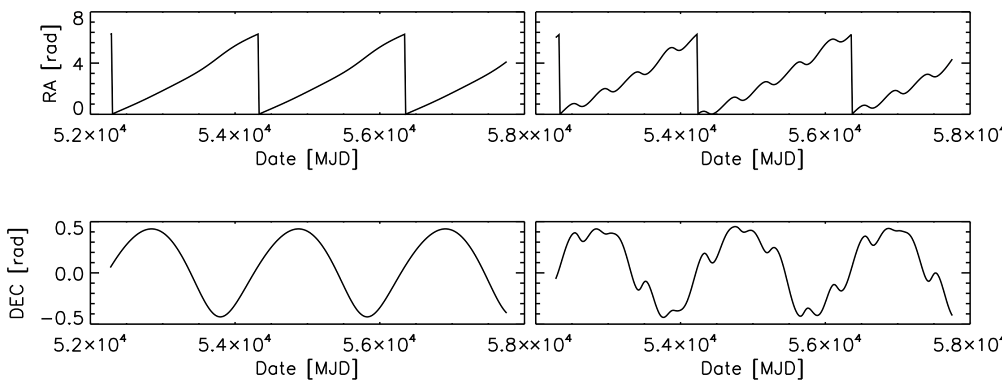

Figure 7: *Left:* The heliocentric equatorial position for Hygiea. The time runs from 2002 Jan 1 to 2016 Dec 31, or an interval of 15 years. The orbital period is 5.56 years. *Right:* Geocentric ecliptic position.

The right hand panel of Figure 7 shows geocentric equatorial the position. The effect of the motion of the Earth is evident as an annual year ripple superimposed on the overall orbital motion of the asteroid.

Table 5: Orbital elements for 10 Hygeia.

Name	Symbol	Value
Semimajor axis [AU]	a	3.13864
Epoch of perihelion [Julian date]	τ	2455714.653
Argument of perihelion [deg]	ω	313.1924
Longitude of ascending node [deg]	Ω	283.45059
Inclination [deg]	i	3.84215
Eccentricity	e	0.1173

6 The JPL Horizons Ephemeris

The ephemeris information in Table 2 can be generated either using the web interface, or you can send an email to horizons@ssd.jpl.nasa.gov with subject set to `JOB.` The example shown below requests the (X, Y, Z) position vector once a day in the heliocentric, ecliptic coordinate system. The cryptic keywords `COMMAND= '399'` defines the target as the earth, and `CENTER='500@10'` sets the sun as the origin. This request by default also generates the velocity components (V_X, V_Y, V_Z).

```
From: UG Astronomer <ay122@ugastro.berkeley.edu>
Date: November 1, 2008 10:12:52 PM PDT
To: Horizons System Ephemeris <horizons@ssd.jpl.nasa.gov>
```

```
Subject: JOB

!$$SOF
! Comments start with an exclamation point. Don't
! delete the magic start and end strings
EMAIL_ADDR = 'ay122@ugastro.berkeley.edu'
! Add the edmail address you want the response sent to
COMMAND    = '399'
! Object 399 is the earth
OBJ_DATA   = 'NO'
! Don't print the summary data for the earth
TABLE_TYPE = 'VECTORS'
! Return (X,Y,Z) and (VX,VY,VZ)
REF_PLANE  = 'ECLIPTIC'
! Ecliptic coordinates
MAKE_EPHEM = 'YES'
! Return the computation
CENTER     = '500@10'
! Coordinate systems is centered on the sun
START_TIME = '2008-AUG-24 0:00'
STOP_TIME  = '2008-AUG-26 0:00'
STEP_SIZE  = '1 day'
! Start, stop, and interval
REF_SYSTEM = 'J2000'
! Equinox is 2000.0
VEC_LABELS = 'NO'
! Don't print X=, Y=, Z=, &c.
OUT_UNITS  = 'AU-D'
! Use AU and days instead of km and km/s
!$$EOF
```

7 References

The treatment is based primarily on Ch. 6 & 7 of “Spherical Astronomy”, Robin M. Green, Cambridge, 1985. The major difference is that derivations using spherical trigonometry have been replaced with rotation matrices (yeah!).