Chapter 7

The Orbital Elements of a Visual Binary Star

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The True Orbit

Whilst astronomers regard the brighter component as fixed and map the motion of the fainter one around it, in reality, both stars in a binary system move in ellipses around the common centre of gravity. The size of the ellipse is directly proportional to the mass of the star, so in the Sirius system, for instance, the primary has a mass of 1.5 $M_{\odot}$, the white dwarf companion 1.0 $M_{\odot}$ and so the size of ellipses traced out on the sky are in the ratio 1.0 to 1.5 for the primary and secondary (Figure 7.1). The ratio of the masses is inversely proportional to the size of the apparent orbits (see eqn 1.1 in Chapter 1), so this gives one relation between the two masses. To get the sum of the masses requires the determination of the true orbit from the apparent orbit and this is what this chapter will describe.

We regard the primary star as fixed and measure the motion of the secondary star with respect to it, and in Chapter 1 we saw that in binary stars the motion of the secondary star with respect to the primary is an ellipse. This is called the apparent ellipse or orbit and is the projection of the true orbit on the plane of the sky. Since the eccentricities of true orbits can vary from circular to extremely elliptical (in practice the highest eccentricity so far observed is 0.975), then the range of apparent ellipses is even more varied because the true orbit can be tilted in two dimensions at any angle to the line of sight. We need the true orbit in order to determine the sum of the masses of the two stars in the

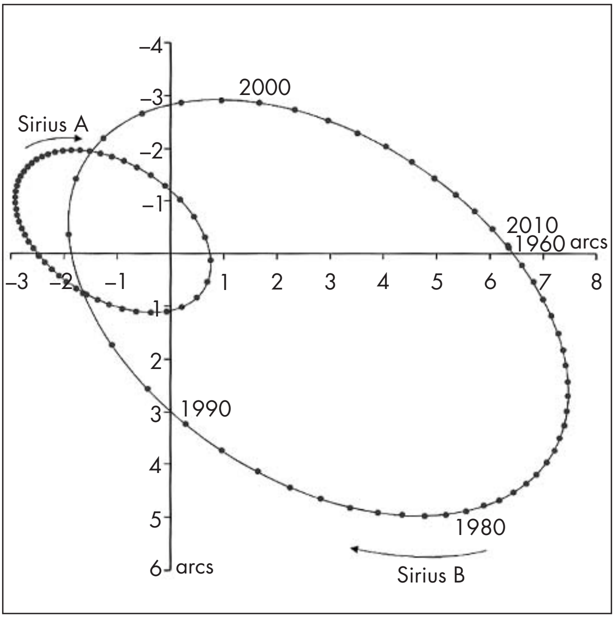

Figure 7.1. The real orbits of the stars in the Sirius system.

binary. This is still the only direct means of finding stellar masses.

On the face of it then the measurements that we make of separation and position angle at a range of epochs are all the information that we have to try and disentangle the true orbit from the apparent orbit. We do, however also know the time at which each observation was made much more precisely than either of the measured quantities. There are other clues, for instance, in the way that the companion moves in the apparent orbit.

In Figure 7.2 I plot the apparent motion of the binary OΣ 363. In this case (x, y) rectangular coordinates are used rather than the θ, ρ polar coordinates which are more familiar to the observer. Each dot on the apparent ellipse represents the position of the companion at two-year intervals. It is immediately clear that the motion is not uniform but it is considerably faster in the third quadrant i.e. between south and west. The point at which the motion is fastest represents the periastron (or closest approach) in *both* the true and the apparent orbits.

Kepler's second law tells us that areas swept out in given times must be equal and this also applies to both the true and the apparent orbit. In Figure 7.2 although the three shaded areas are shown at different points in

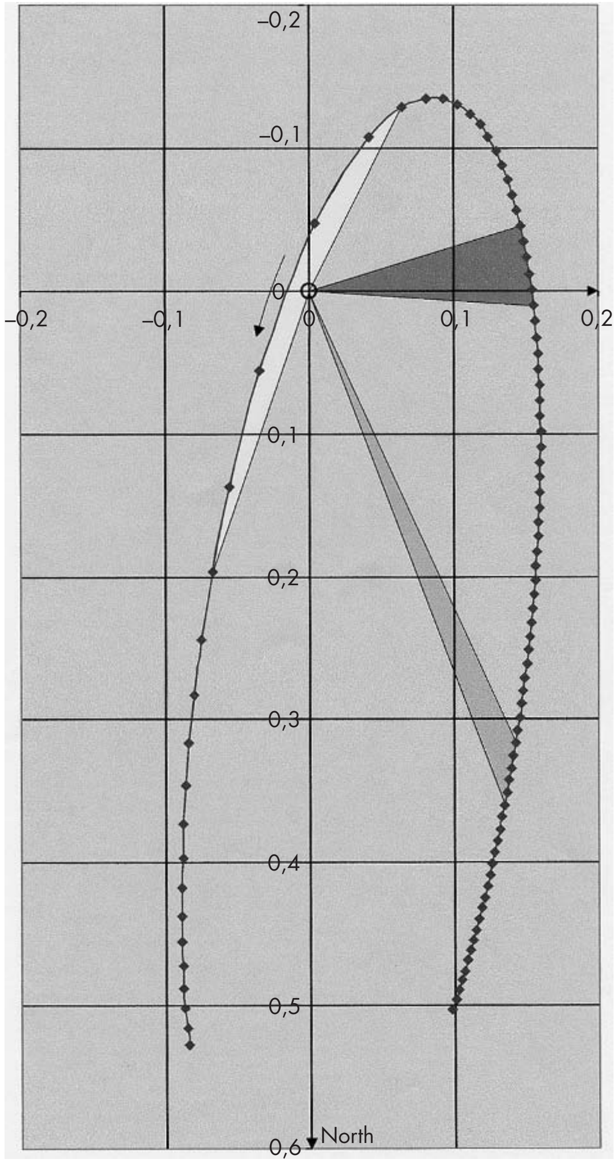

Figure 7.2. The apparent orbit of a visual binary star.

the apparent orbit because they are all traced out over a ten-year interval, the areas are the same. We also know that the centre of the apparent orbit is the projected centre of the true orbit. In most cases the motion is described by the fainter star relative to the brighter star that is fixed in the focus of the ellipse as if the total mass were concentrated in the fixed centre of attraction.

In the true orbit the centre of the ellipse is called C, the focus, and where the brighter star is located is called

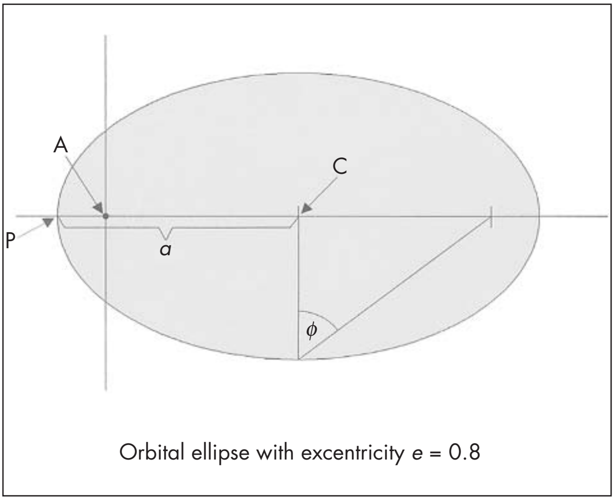

Figure 7.3. The true elements of a visual binary star.

A. The periastron P is the closest point of the ellipse to A. The geometry of the motion suggests use of polar coordinates. The elements of the real orbit are as follows (Figure 7.3):

P the revolution period in years; alternatively the mean motion
per year ($n = 360/P$ or $\mu = 2\pi/P$ is given);

T the time passage through periastron;

e the numerical eccentricity *e* of the orbit; the auxiliary angle ϕ is given by $e = \sin\phi$;

a the semiaxis major in arcseconds.

The Apparent Orbit

The apparent (observed) orbit results from a projection of the true orbit onto the celestial sphere (Figure 7.4). Three more elements determine this projection:

Ω the position angle of the ascending node. This is the position angle of the line of intersection between the plane of projection and the true orbital plane. The angle is counted from north to the line of nodes. The ascending node is the node where the motion of the companion is directed away from the Sun. It differs from the second node by 180° and can be determined only by radial-velocity measurements. If the ascending node is unknown, the value < 180° is given.

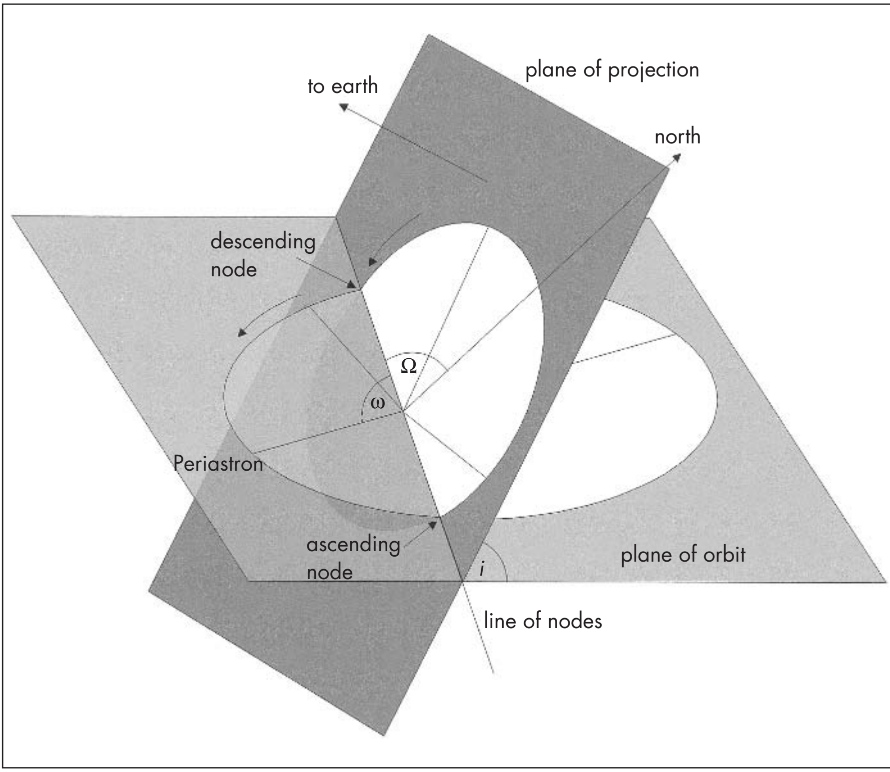

Figure 7.4. The projected elements of a visual binary star.

i the orbital inclination. This is the angle between the plane of projection and the true orbital plane. Values range from 0° to 180°. For $0° \leq i < 90°$ the motion is called direct. The companion then moves in the direction of increasing position angles (anticlockwise). For $90° < i \leq 180°$ the motion is called retrograde.

ω the argument of periastron. This is the angle between the node and the periastron, measured in the plane of the true orbit and in the direction of the motion of the companion.

The elements P, T, a, e, i, ω, Ω, are called the Campbell elements. There is another group of elements which is used in order to calculate rectangular coordinates. They are called Thiele–Innes elements (Figure 7.5):

$$A = a\,(\cos\omega\ \cos\Omega - \sin\omega\ \sin\Omega\ \cos i)$$
$$B = a\,(\cos\omega\ \sin\Omega + \sin\omega\ \cos\Omega\ \cos i)$$

$$F = a\,(-\sin\omega\ \cos\Omega - \cos\omega\ \sin\Omega\ \cos i)$$
$$G = a\,(-\sin\omega\ \sin\Omega + \cos\omega\ \cos\Omega\ \cos i).$$

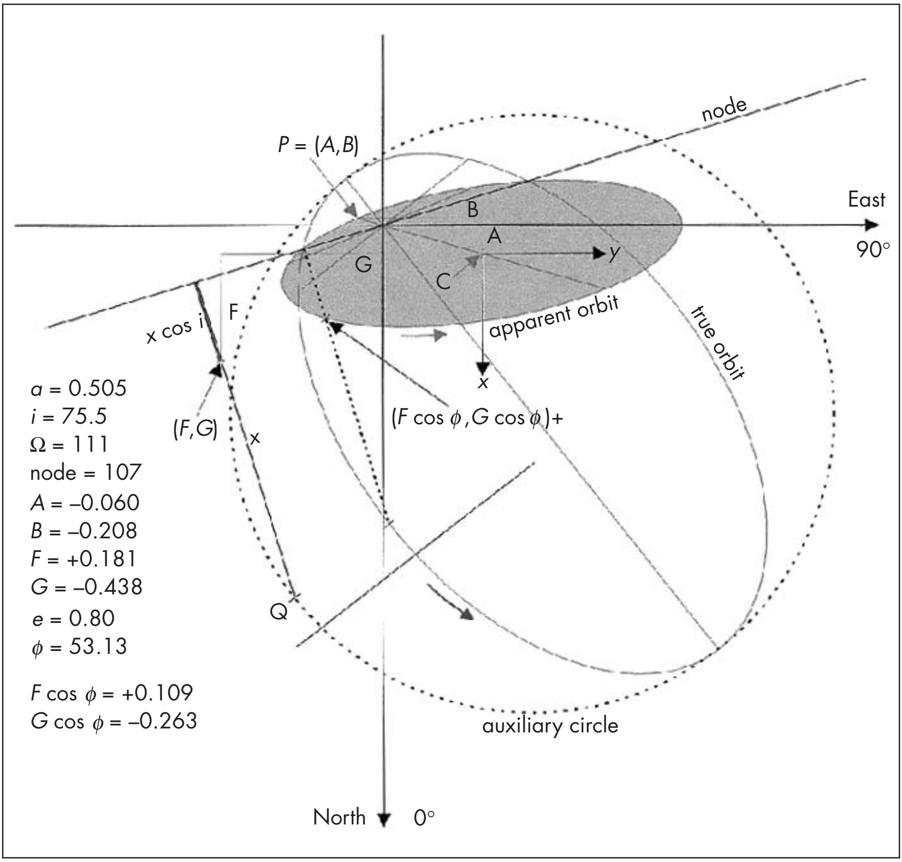

Figure 7.5. Thiele–Innes elements and Campbell elements.

Note that the elements A, B, F and G are independent of the eccentricity *e*. The points (*A*, *B*) and (*F* cos ϕ, *G* cos ϕ), together with the centre of the apparent ellipse, define a pair of conjugate axes which are the projections of the major and minor axes of the true orbit.

There is an instructive and easy way to draw the apparent orbit from the seven Campbell elements. It runs as follows:

1. Draw the rectangular coordinate system with a convenient scale. North is at the bottom (the positive *x*-axis); east is at the right (the positive *y*-axis).
2. Draw the line of nodes. The node makes the angle Ω between north and the line of nodes.
3. Lay off the angle ω from the line of nodes and proceeding in the direction of the companion's motion, i.e. clockwise, when $i > 90°$, and counterclockwise,

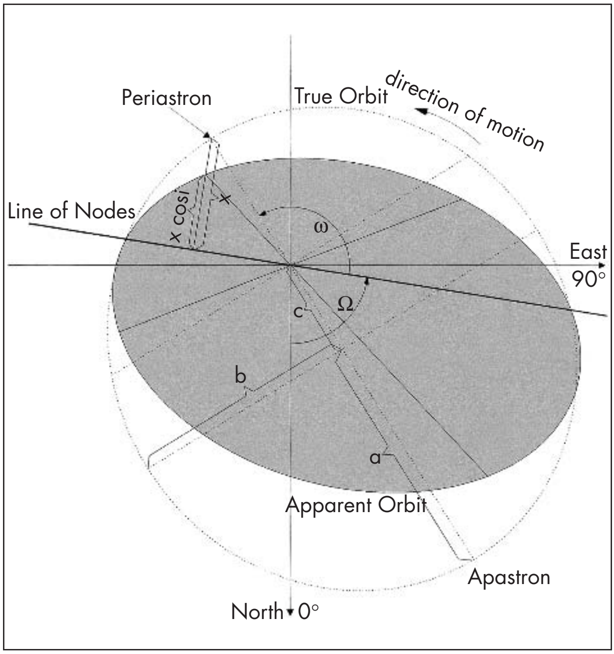

Figure 7.6. The true and the projected orbit of OΣ 235 drawn in one plane. Note: the law of areas holds in the projected ellipse as well.

when $i < 90°$. This will give the line of periastron and apastron of the true orbit.

4. Draw the true orbit ellipse. The distance of the centre of the true orbit from the centre of the coordinate system is c. The long axis is $2a$, the short axis is $2b$, so b and c are easily calculated:

$$c = ae; b = \sqrt{a^2 - c^2}$$

5. Construct the apparent orbit. Draw lines from points on the true orbit to the line of nodes; the lines have to be perpendicular to the line of nodes. Multiply the lines by cos i. Connecting the so obtained points yields the apparent orbit.

As an example, the orbit for OΣ 235 is given in Figure 7.6. Elements are as follows (Heintz[1]): P = 73.03 years, T = 1981.69, a = 0″.813, e = 0.397, i = 47°.3, ω = 130°.9, Ω = 80°.9.

Ephemeris Formulae

For any time t, the coordinates θ, ρ or x, y are computed from the elements by means of the following formulae. The auxiliary circle has radius a. See Figure 7.7,

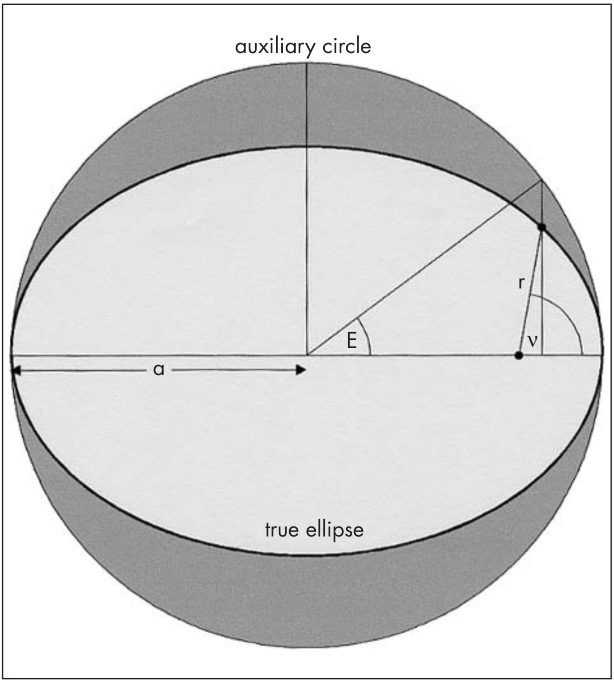

Figure 7.7. Auxiliary circle, eccentric anomaly *E*, true anomaly *v* and radius vector *r*.

The angle E is called the eccentric anomaly and has to be determined from the mean anomaly M:

$$\mu\,(t - T) = M = E - \sin E \quad \text{(Kepler's equation).}$$

This equation is transcendental, i.e. it is not algebraic and has to be solved iteratively. A first approximation is given by the formula:

$$E_0 = M + e \sin M + \frac{e^2}{M} \sin 2M$$

This new E_0 is used to calculate a new M_0:

$$M_0 = E_0 - e \sin E_0$$

A new E_1 is obtained from M, M_0 and E_0:

$$E_1 = E_0 + \frac{M - M_0}{1 - e \cos\ E_0}$$

The last two formulae are iterated to the desired accuracy. Four iterations are sufficient for $e \leq 0.95$. Now the desired positions are calculated:

Polar coordinates:

$$\tan\frac{\nu}{2}=\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$$

$$r=\frac{a(1-e^2)}{1+e\cos\nu}$$

$$\tan(\theta-\Omega)=\tan(\nu+\omega)\cos i$$

$$\rho=r\cos(\nu+\omega)\sec(\theta-\Omega).$$

An alternative formula for the calculation of ρ, due to Michael Greaney,[2] obviates the possibility of the formula becoming undefined, e.g. when $\theta-\Omega=90°$:

$$y=\sin(\theta-\Omega)\cos i$$

$$x=\cos(\theta-\Omega)$$

$$\tan(\theta-\Omega)=\frac{y}{x}$$

$$\rho=r\sqrt{x^2+y^2}.$$

Rectangular coordinates:

$$X=\cos E-e;\qquad Y=\sqrt{1-e^2}\sin E$$

$$x=AX+FY;\qquad y=BX+GY.$$

References

1 Heintz, W.D., 1990, *Astron. Astrophys. Suppl.*, **82**, 65.

2 Greaney, M.P., 1997, Calculating separation from binary orbits: an alternative expression, *Webb Society Quarterly Journal*, **107**.