Convolution, Correlation, & Fourier Transforms

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Introduction

- A large class of signal processing techniques fall under the category of ***Fourier transform*** methods
  - These methods fall into two broad categories
    - Efficient method for accomplishing common data manipulations
    - Problems related to the Fourier transform or the power spectrum

Time & Frequency Domains

- A physical process can be described in two ways
  - In the *time domain*, by h as a function of time t, that is $h(t)$, $-\infty < t < \infty$
  - In the *frequency domain*, by H that gives its amplitude and phase as a function of frequency f, that is $H(f)$, with $-\infty < f < \infty$
    - In general h and H are complex numbers
- It is useful to think of $h(t)$ and $H(f)$ as two different representations of the same function
  - One goes back and forth between these two representations by Fourier transforms

Fourier Transforms

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t}\, dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{2\pi i f t}\, df$$

- If t is measured in seconds, then f is in cycles per second or Hz
- Other units
  - E.g, if $h=h(x)$ and x is in meters, then H is a function of spatial frequency measured in cycles per meter

Fourier Transforms

- The Fourier transform is a linear operator
  - The transform of the sum of two functions is the sum of the transforms

$$h_{12} = h_1 + h_2$$

$$H_{12}(f) = \int_{-\infty}^{\infty} h_{12} e^{-2\pi i f t}\, dt$$

$$= \int_{-\infty}^{\infty} (h_1 + h_2) e^{-2\pi i f t}\, dt = \int_{-\infty}^{\infty} h_1 e^{-2\pi i f t}\, dt + \int_{-\infty}^{\infty} h_2 e^{-2\pi i f t}\, dt$$

$$= H_1 + H_2$$

Fourier Transforms

- $h(t)$ may have some special properties
  - Real, imaginary
  - Even: $h(t) = h(-t)$
  - Odd: $h(t) = -h(-t)$
- In the frequency domain these symmetries lead to relations between $H(f)$ and $H(-f)$

FT Symmetries

If…	Then…
$h(t)$ real	$H(-f) = [\ H(f)\ ]^*$
$h(t)$ imaginary	$H(-f) = -[\ H(f)\ ]^*$
$h(t)$ even	$H(-f) = H(f)$ (even)
$h(t)$ odd	$H(-f) = -\ H(f)$ (odd)
$h(t)$ real & even	$H(f)$ real & even
$h(t)$ real & odd	$H(f)$ imaginary & odd
$h(t)$ imaginary & even	$H(f)$ imaginary & even
$h(t)$ imaginary & odd	$H(f)$ real & odd

Elementary Properties of FT

$$h(t) \leftrightarrow H(f)$$ Fourier Pair

$$h(at) \leftrightarrow \frac{1}{a} H(f/a)$$ Time scaling

$$h(t - t_0) \leftrightarrow H(f) e^{-2\pi i f t_0}$$ Time shifting

Convolution

- With two functions *h(t)* and *g(t)*, and their corresponding Fourier transforms *H(f)* and *G(f)*, we can form two special combinations
  - The *convolution*, denoted $f = g * h$, defined by

$$f(t) = g * h \equiv \int_{-\infty}^{\infty} g(\tau) h(t - \tau) d\tau$$

Convolution

- $g*h$ is a function of time, and

$$g*h = h*g$$

  – The convolution is one member of a transform pair

$$g * h \leftrightarrow G(f)H(f)$$

- The Fourier transform of the convolution is the product of the two Fourier transforms!
  – This is the **Convolution Theorem**

Correlation

- The *correlation* of g and h

$$Corr(g,h) \equiv \int_{-\infty}^{\infty} g(\tau + t)h(\tau)d\tau$$

- The correlation is a function of t, which is known as the lag
  - The correlation lies in the time domain

Correlation

- The correlation is one member of the transform pair

$$Corr(g,h) \leftrightarrow G(f)H^{*}(f)$$

  - More generally, the RHS of the pair is $G(f)H(-f)$
  - Usually g & h are real, so $H(-f) = H^*(f)$
- Multiplying the FT of one function by the complex conjugate of the FT of the other gives the FT of their correlation
  - This is the **Correlation Theorem**

Autocorrelation

- The correlation of a function with itself is called its *autocorrelation*.
  - In this case the correlation theorem becomes the transform pair

$$Corr(g,g) \leftrightarrow G(f)G^*(f) = |G(f)|^2$$

  - This is the **Wiener-Khinchin Theorem**

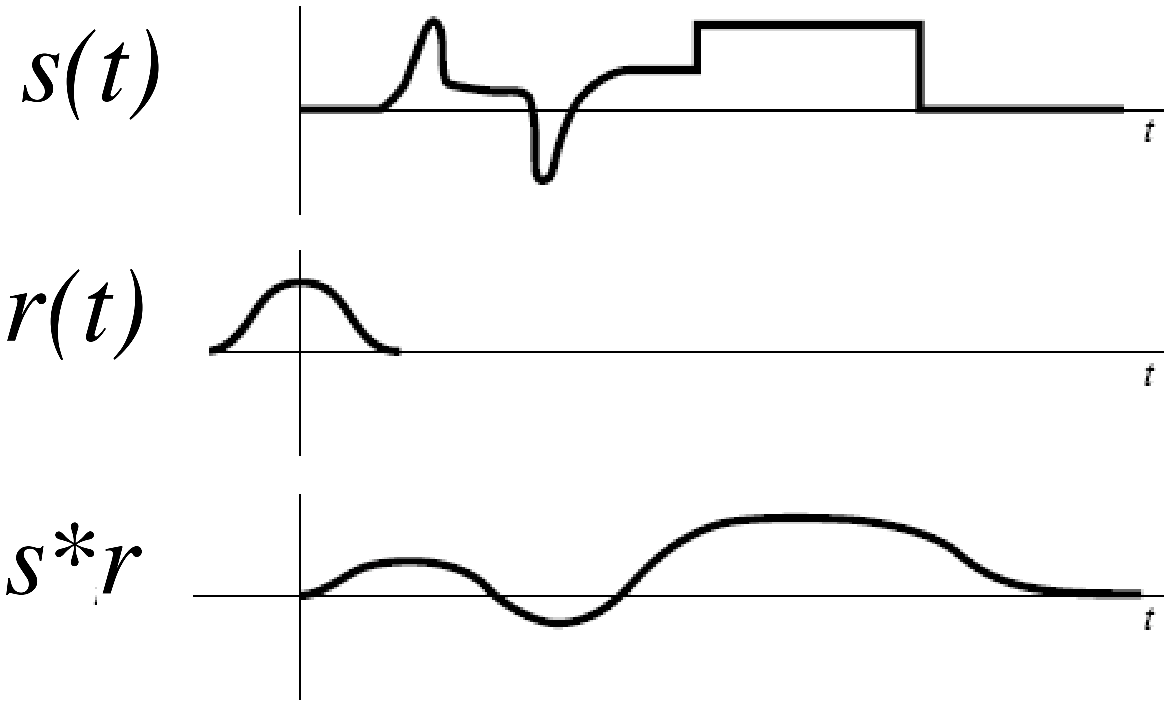
Convolution

- Mathematically the convolution of *r(t)* and *s(t)*, denoted $r*s=s*r$
- In most applications *r* and *s* have quite different meanings
  - *s(t)* is typically a signal or data stream, which goes on indefinitely in time
  - *r(t)* is a response function, typically a peaked and that falls to zero in both directions from its maximum

The Response Function

- The effect of convolution is to smear the signal $s(t)$ in time according to the recipe provided by the response function $r(t)$
- A spike or delta-function of unit area in s which occurs at some time t_0 is
  - Smeared into the shape of the response function
  - Translated from time 0 to time t_0 as $r(t - t_0)$

Convolution



- The signal *s(t)* is convolved with a response function *r(t)*
  - Since the response function is broader than some features in the original signal, these are smoothed out in the convolution

Fourier Transforms & FFT

- Fourier methods have revolutionized many fields of science & engineering
  - Radio astronomy, medical imaging, & seismology
- The wide application of Fourier methods is due to the existence of the **fast Fourier transform** (FFT)
- The FFT permits rapid computation of the discrete Fourier transform
- Among the most direct applications of the FFT are to the convolution, correlation & autocorrelation of data

The FFT & Convolution

- The convolution of two functions is defined for the continuous case
  - The convolution theorem says that the Fourier transform of the convolution of two functions is equal to the product of their individual Fourier transforms

$$g * h \leftrightarrow G(f)H(f)$$

- We want to deal with the discrete case
  - How does this work in the context of convolution?

Discrete Convolution

- In the discrete case *s(t)* is represented by its sampled values at equal time intervals s_j
- The response function is also a discrete set r_k
  - r_0 tells what multiple of the input signal in channel *j* is copied into the output channel *j*
  - r_1 tells what multiple of input signal *j* is copied into the output channel $j+1$
  - r_{-1} tells the multiple of input signal *j* is copied into the output channel $j-1$
  - Repeat for all values of *k*

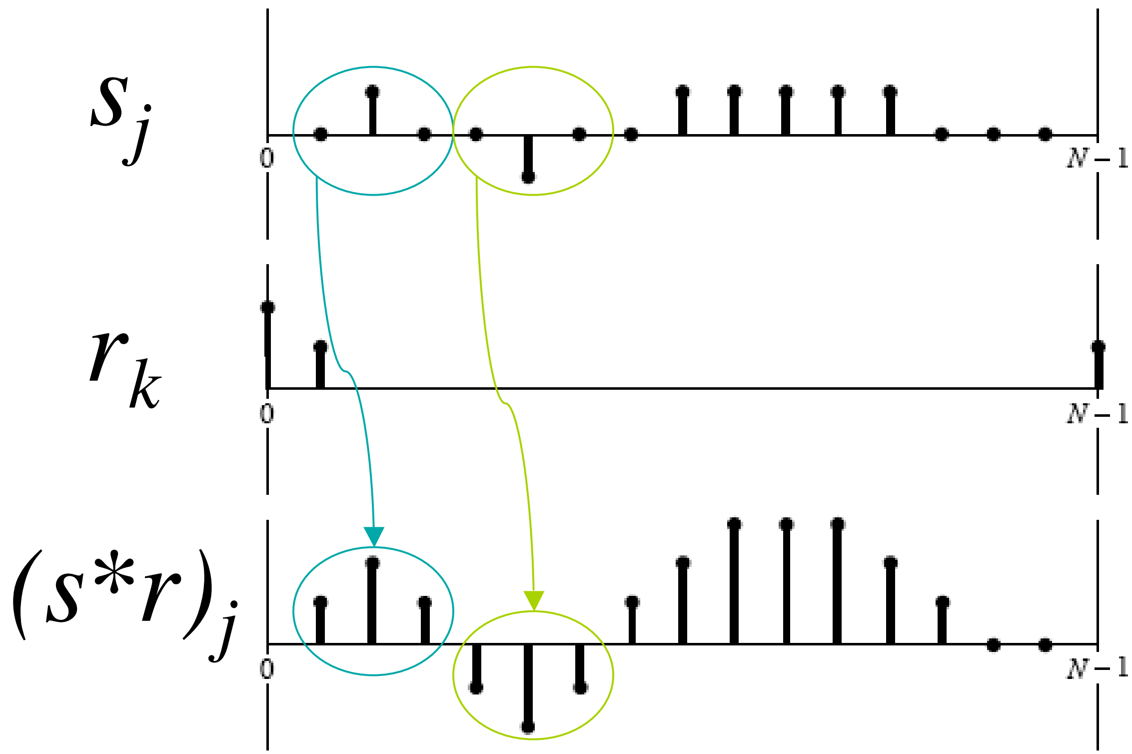
Discrete Convolution

- Symbolically the discrete convolution is with a response function of finite duration, N, is

$$(s * r)_j = \sum_{k=-N/2+1}^{N/2} s_k r_{j-k}$$

$$(s * r)_j \leftrightarrow S_l R_l$$

Discrete Convolution



- Convolution of discretely sampled functions
  - Note the response function for negative times wraps around and is stored at the end of the array r_k

Examples

- Java applet demonstrations
  - Continuous convolution
    - http://www.jhu.edu/~signals/convolve/
  - Discrete convolution
    - http://www.jhu.edu/~signals/discreteconv/

Suppose that f and g are functions of time

$$f(t) = \int_{-\infty}^{\infty} F(\nu)\, e^{2\pi i \nu t} d\nu \quad \text{and} \quad g(t) = \int_{-\infty}^{\infty} G(\nu)\, e^{2\pi i \nu t} d\nu$$

The convolution f*g says

$$f * g = \int_{-\infty}^{\infty} g(t')f(t - t')dt'$$

$$= \int_{-\infty}^{\infty} g(t')\left[\int_{-\infty}^{\infty} F(\nu)\, e^{2\pi i \nu (t-t')} d\nu \right] dt'$$

Swap the order of integration

$$= \int_{-\infty}^{\infty} F(\nu)\left[\int_{-\infty}^{\infty} g(t')\, e^{-2\pi i \nu t'}\, dt' \right] e^{2\pi i \nu t} d\nu$$

$$= FT\left[F(\nu)G(\nu)\right]$$

Voila!