A Little Error Propagation: finding the error in $\langle x \rangle$

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The Centroid

The centroid of a star is the "center of light"

$$\langle x \rangle = \sum_i x_i I_i \Big/ \sum_i I_i$$

and the, s, the rms width of the stellar image is

$$s^2 = \sum_i \left(x_i - \langle x \rangle \right)^2 I_i \Big/ \sum_i I_i = \langle x^2 \rangle - \langle x \rangle^2$$

but what's the error in $\langle x \rangle$…

Error propagation

If $y = f(x_1, x_2, x_3\ldots)$ The fundamental law of error propagation is

$$\sigma_y^2 = \sigma_{x_1}^2\left(\frac{\partial f}{\partial x_1}\right)^2 + \sigma_{x_2}^2\left(\frac{\partial f}{\partial x_2}\right)^2 + \sigma_{x_3}^2\left(\frac{\partial f}{\partial x_3}\right)^2 + \ldots$$

For a quantity where the errors in $x_1, x_2\ldots$ are uncorrelated

If we apply this to the formula for $\langle x \rangle$

$$\sigma_{\langle x \rangle}^2 = \sigma_{I_1}^2 \left(\frac{\partial \langle x \rangle}{\partial I_1} \right)^2 + \sigma_{I_2}^2 \left(\frac{\partial \langle x \rangle}{\partial I_2} \right)^2 + \sigma_{I_3}^2 \left(\frac{\partial \langle x \rangle}{\partial I_3} \right)^2 + \ldots$$

$$= \sum_i \sigma_{I_i}^2 \left(\frac{\partial \langle x \rangle}{\partial I_i} \right)^2$$

So the tricky part is computing

$$\frac{\partial \langle x \rangle}{\partial I_j} = \frac{\partial}{\partial I_j}\left(\frac{\sum_i x_i I_i}{\sum_i I_i} \right)$$

Using the chain rule

$$\frac{\partial\langle x\rangle}{\partial I_j} = \frac{\partial}{\partial I_j}\left(\frac{\sum_i x_i I_i}{\sum_i I_i}\right)$$

$$= \sum_i x_i I_i (-1)\left(\sum_i I_i\right)^{-2} \frac{\partial}{\partial I_j}\left(\sum_i I_i\right) + \left(\sum_i I_i\right)^{-1} \frac{\partial}{\partial I_j}\left(\sum_i x_i I_i\right)$$

$$= -\frac{\sum_i x_i I_i}{\left(\sum_i I_i\right)^2}\delta_{ij} + \frac{1}{\sum_i I_i} x_i \delta_{ij}$$

$$= \frac{1}{\sum_i I_i}\left(x_j - \langle x\rangle\right)$$

Putting it all together

$$\sigma_{\langle x\rangle}^2 = \sum_j \sigma_{I_j}^2 \left(\frac{\partial\langle x\rangle}{\partial I_j}\right)^2$$

$$= \sum_j \sigma_{I_j}^2 \left(\frac{1}{\sum_i I_i}\left(x_j - \langle x\rangle\right)\right)^2$$

$$= \sum_j \sigma_{I_j}^2 \left(x_j - \langle x\rangle\right)^2 \Big/ \left(\sum_i I_i\right)^2$$

For Poisson statistics, *i.e.*, if I_i is measured in photoelectrons

$$\sigma_{\langle x \rangle}^2 = \sum_j I_j \left(x_j - \langle x \rangle \right)^2 \Bigg/ \left(\sum_i I_i \right)^2$$

Finally, rewrite the error in x in terms of the stellar width $s = (\langle x^2 \rangle - \langle x \rangle^2)^{1/2}$ and the total number of detected photoelectrons, $F = \sum I_i$,…

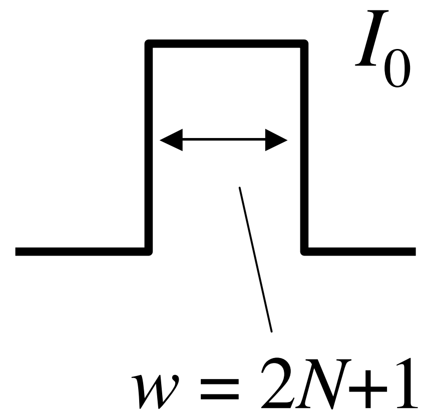
$$\sigma_{\langle x \rangle}^2 = \sum_j I_j \left(x_j - \langle x \rangle\right)^2 \Big/ \left(\sum_i I_i\right)^2$$

or referring back to the definition of s on p. 2

$$\sigma_{\langle x \rangle}^2 = s^2 / F$$

An simple example

- Suppose the star image is a "top hat" shape so that $I_i = I_0$ for i = -N, …-1, 0, 1, …, N
  - Total flux is $F = I_0(2N+1)$
  - The star is centered so that $\langle x \rangle = 0$



$$\sigma^2_{\langle x \rangle} = \left(\sum_{i=-N}^{N} I_i \right)^{-2} \sum_{j=-N}^{N} I_j x_j^2$$

$$= \frac{1}{3} \frac{N(N+1)}{F}$$

Evidently $\sigma \propto F^{-1/2}$ and precision decreases in proportion to w