Fourier Transform of Discretely Sampled Data

James R. Graham

11/28/2009

Introduction

- A large class of signal processing techniques fall under the category of ***Fourier transform*** methods
  - These methods fall into two broad categories
    - Efficient method for accomplishing common data manipulations
    - Problems related to the Fourier transform or the power spectrum

Time & Frequency Domains

- A physical process can be described in two ways
  - In the *time domain*, by the values of some some quantity h as a function of time t, that is $h(t)$, $-\infty < t < \infty$
  - In the *frequency domain,* by the complex number, H, that gives its amplitude and phase as a function of frequency f, that is $H(f)$, with $-\infty < f < \infty$
- It is useful to think of $h(t)$ and $H(f)$ as two different representations of the same function
  - One goes back and forth between these two representations by Fourier transforms

Fourier Transforms

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t} \, dt$$

$$h(t) = \int_{-\infty}^{\infty} H(f) e^{2\pi i f t} \, df$$

- If t is measured in seconds, then f is in cycles per second or Hz
- Other units
  - E.g, if $h=h(x)$ and x is in meters, then H is a function of spatial frequency measured in cycles per meter

Discretely Sampled Data

- In the most common situations, function *h(t)* or *h(x)* is ***sampled*** at evenly spaced intervals
  - Δ is the interval between consecutive samples
  - A the sequence of sampled values is

    $$h_n = h(n\Delta),\ n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$
  - $1/\Delta$ is the sampling rate
    - If h is a time series, Δ is in seconds, and the sampling rate is in Hz
    - If h is a spatial series, Δ is in meters, and the sampling rate is in m^{-1}

The Discrete Fourier Transform

- Approximate the integral as a sum

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-2\pi i f t}\, dt \approx \Delta \underbrace{\sum_{k=0}^{N-1} h_k e^{-2\pi i f_n t_k}}_{DFT}$$

and the frequencies are

$$f_n \equiv n/N\Delta, \quad n = -N/2, \ldots, N/2$$

The Sampling Theorem

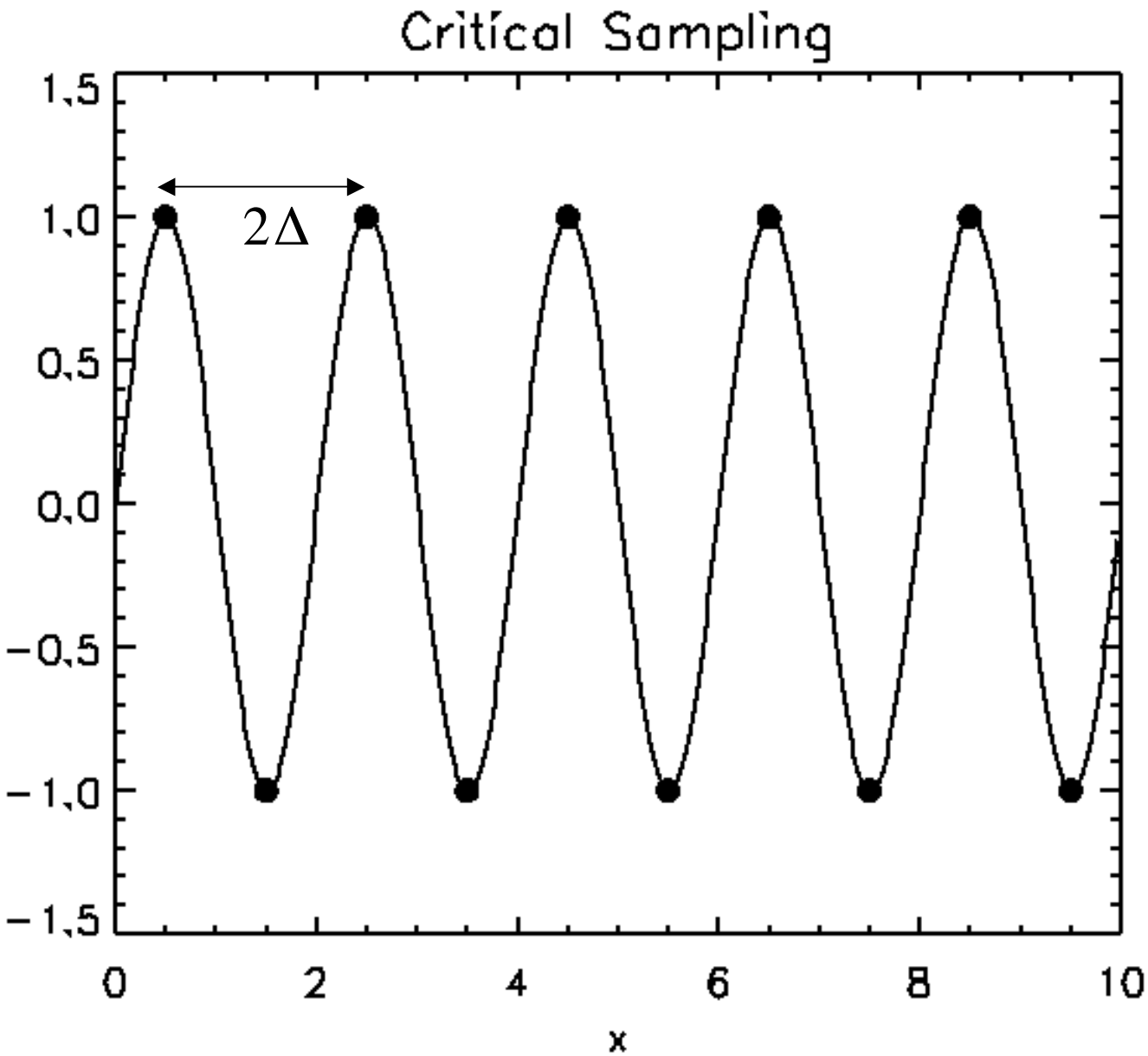
- For any sampling interval Δ, there is a corresponding frequency f_c
- f_c is the ***Nyquist frequency***

$$f_c = 1/\Delta$$

- If a sine wave of the Nyquist frequency is sampled at its positive peak value, then the next sample will be at its negative trough value, the sample after that at the positive peak again, &c.

$f_c = 0.5$

$\Delta = 1$



The Nyquist Frequency

- Critical sampling of a sine wave is two sample points per cycle
- If we choose to measure time (or space) in units of the sampling interval Δ the Nyquist critical frequency is 1/2
- The Nyquist critical frequency is important for two related, but distinct, reasons
  - Good news: Sampling theorem
  - Bad news: Aliasing

The Sampling Theorem

- If a continuous function $h(t)$, sampled at an intervals, Δ, is ***bandwidth limited*** to frequencies smaller in magnitude than f_c

$$H(f) = 0 \text{ for } -f_c \geq f \geq f_c$$

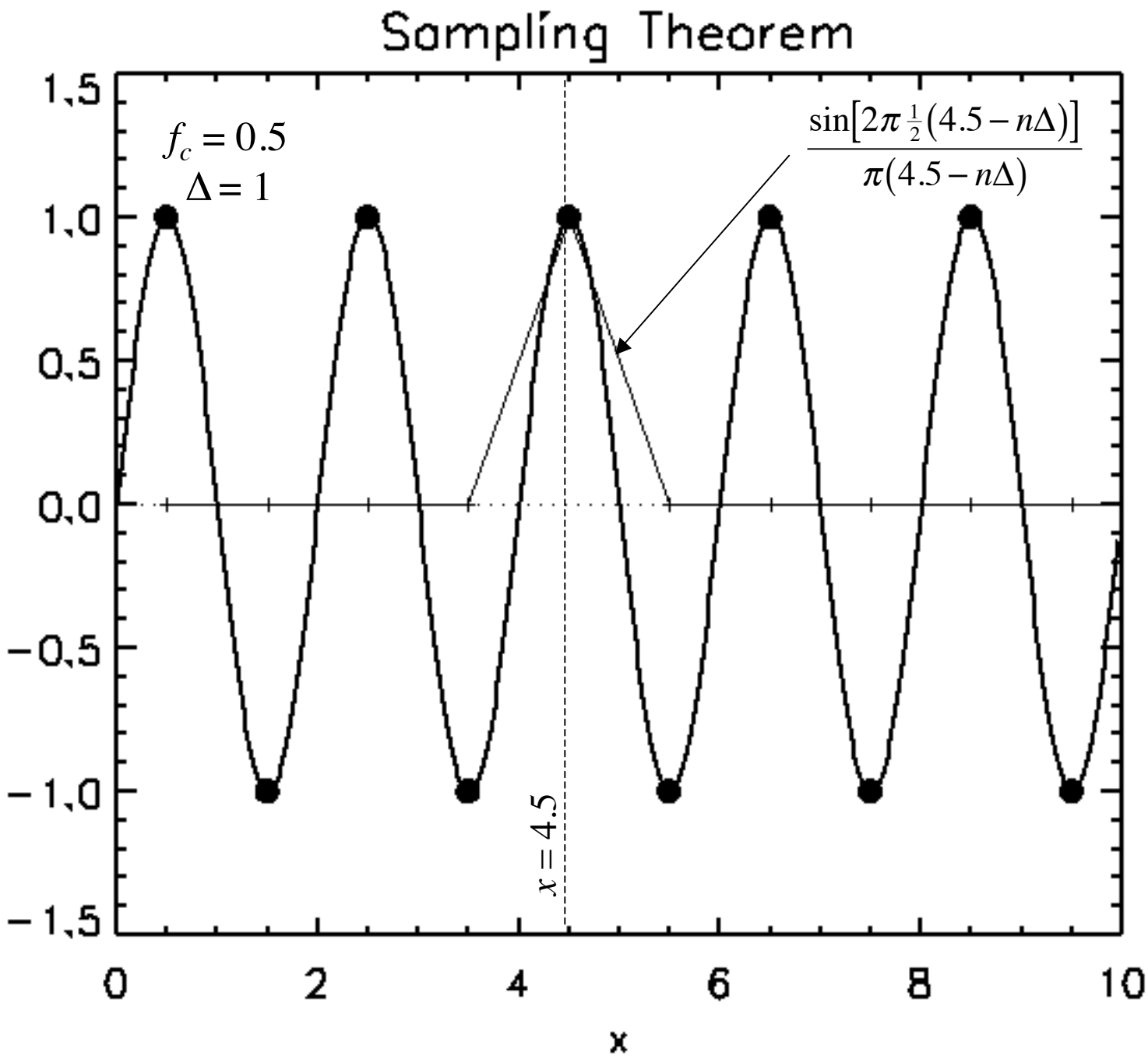
then $h(t)$ is completely determined by the samples h_n

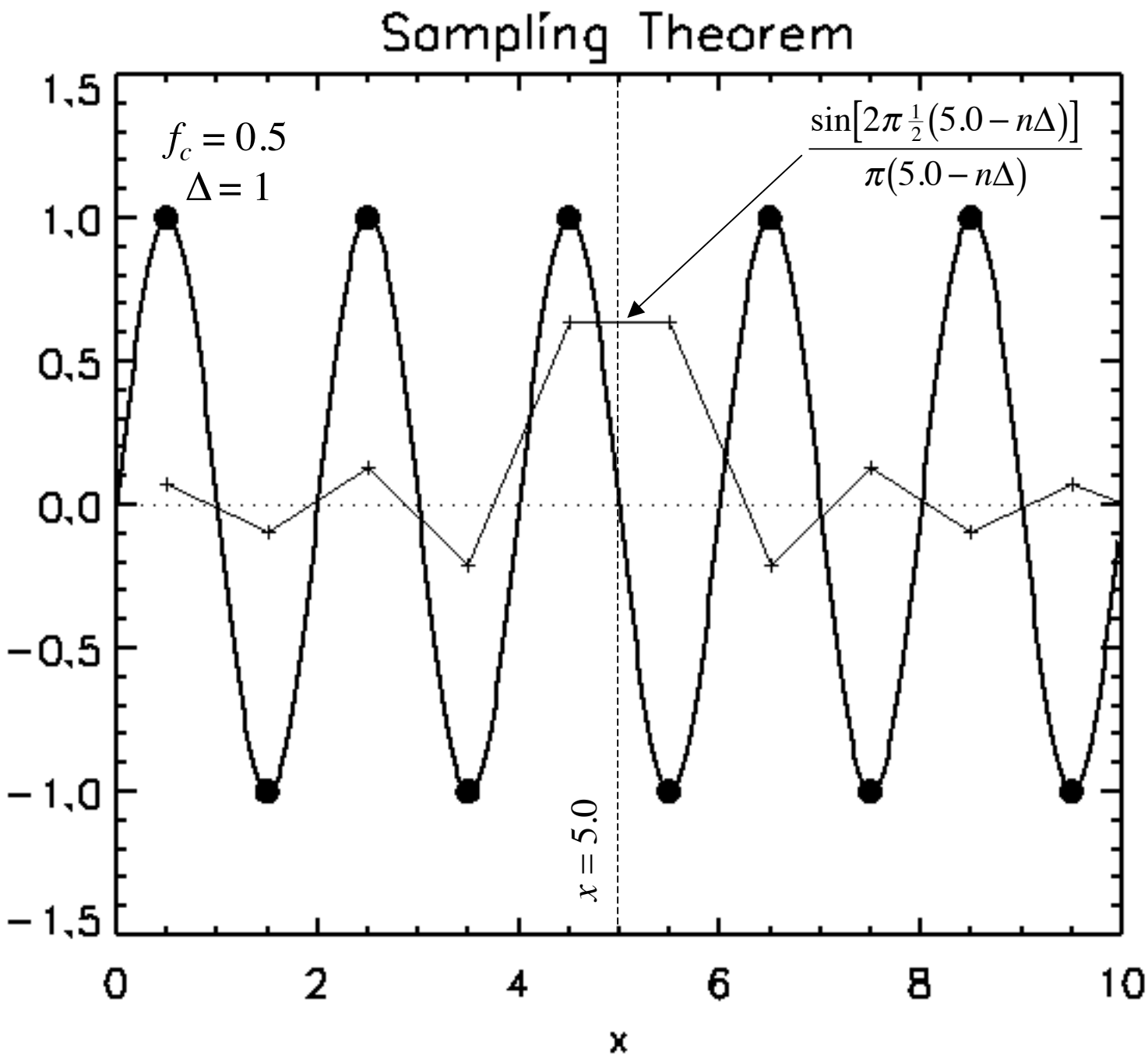
The Sampling Theorem

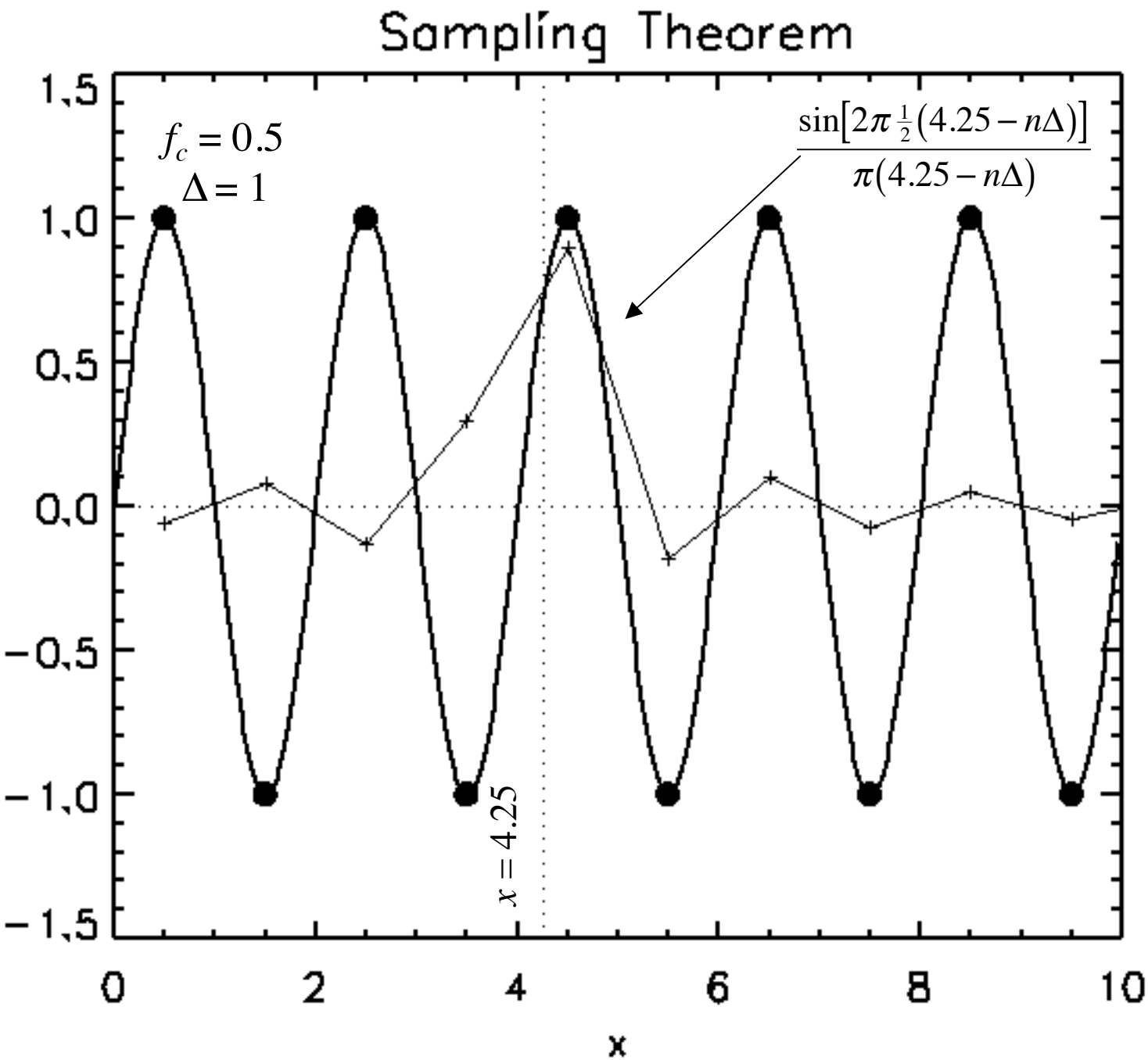
- The the function *h(t)* is given by the formula

$$h(t) = \Delta \sum_{n=-\infty}^{n=+\infty} h_n \frac{\sin\left[2\pi f_c (t - n\Delta)\right]}{\pi (t - n\Delta)}$$

- The continuous function can be reconstructed by convolving the discrete samples with a sinc function


Sampling Theorem
$f_c = 0.5$
$\Delta = 1$
$\frac{\sin[2\pi \frac{1}{2}(4.5 - n\Delta)]}{\pi(4.5 - n\Delta)}$
$x = 4.5$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
2
4
6
8
10
x


Sampling Theorem
$f_c = 0.5$
$\Delta = 1$
$\frac{\sin[2\pi \frac{1}{2}(5.0 - n\Delta)]}{\pi(5.0 - n\Delta)}$
$x = 5.0$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
2
4
6
8
10
x


Sampling Theorem
$f_c = 0.5$
$\Delta = 1$
$\frac{\sin[2\pi \frac{1}{2}(4.25 - n\Delta)]}{\pi(4.25 - n\Delta)}$
$x = 4.25$
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
2
4
6
8
10
x

The Sampling Theorem

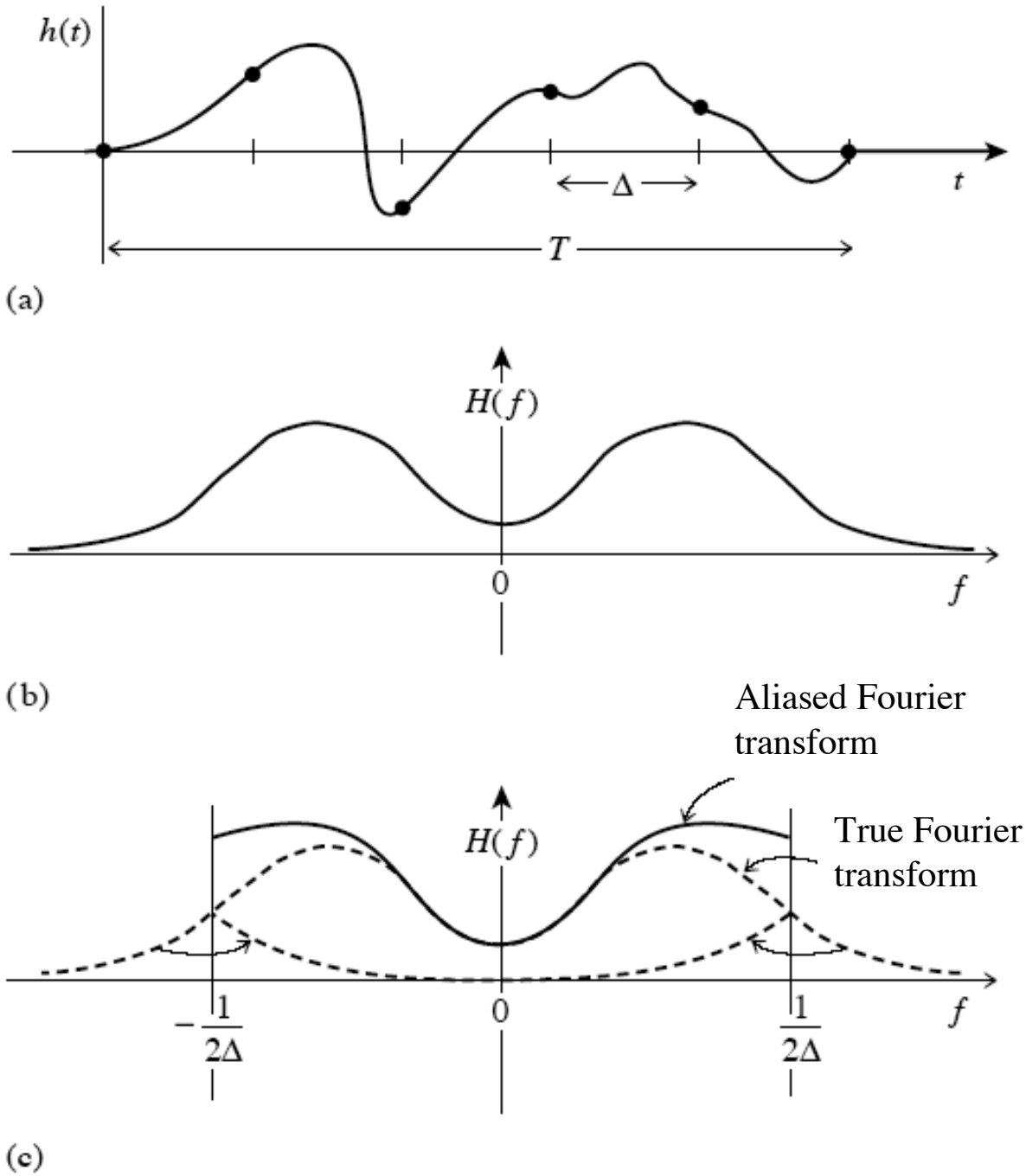
- The sampling theorem is remarkable
  - The "information content" of a bandwidth limited function is, in some sense, infinitely smaller than that of a general continuous function
- Often, signals are known on physical grounds to be bandwidth limited
  - e.g., the signal may have passed through an amplifier with a known, finite frequency response
  - The sampling theorem tells us that the entire information content of the signal can be recorded by sampling it at a rate $1/\Delta$ equal to twice the maximum frequency passed by the amplifier

Aliasing

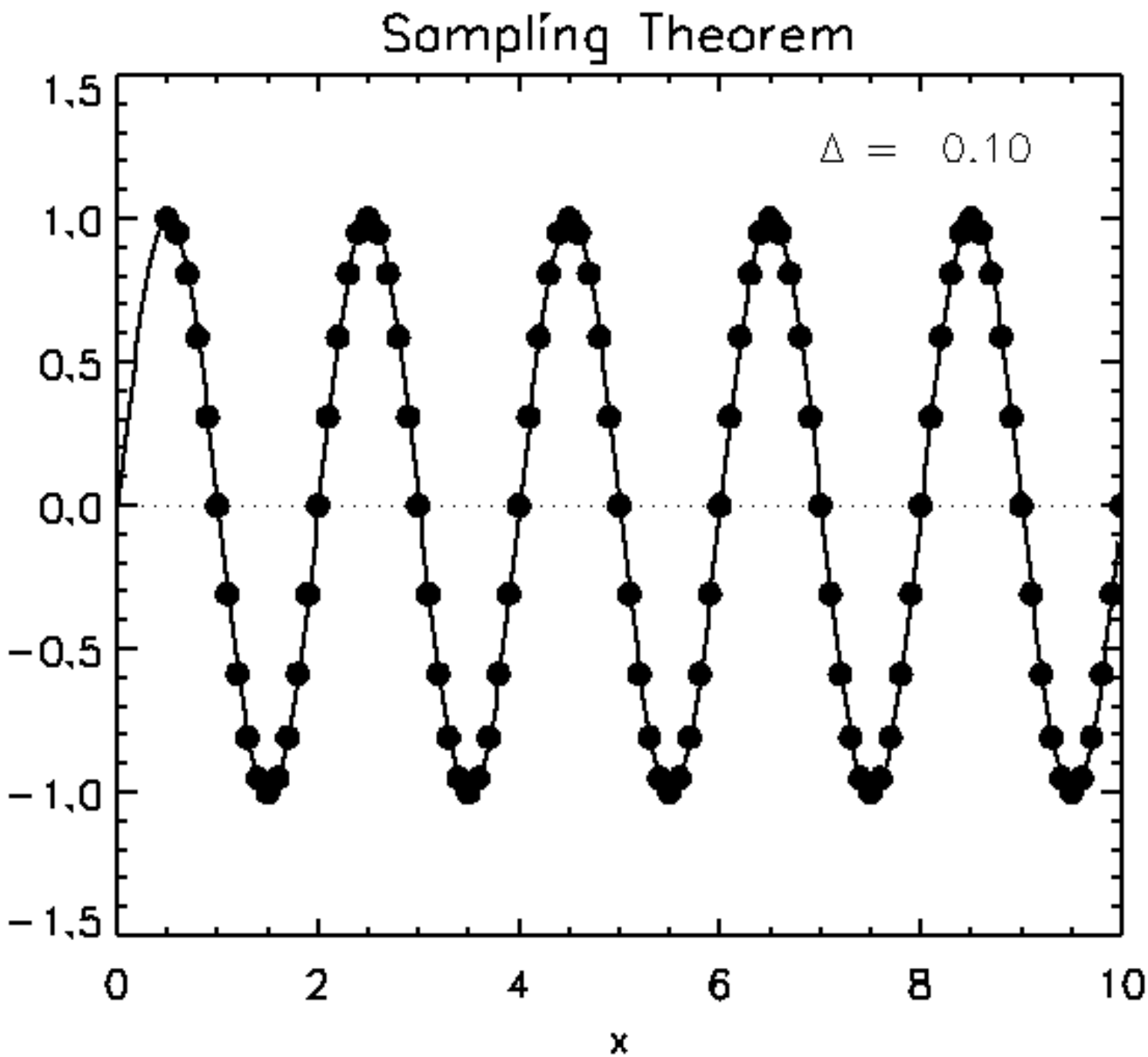
- The bad news
  - What happens when sampling a continuous function that is not bandwidth limited to less than the Nyquist critical frequency?
  - All of the power spectral density that lies outside of the frequency range

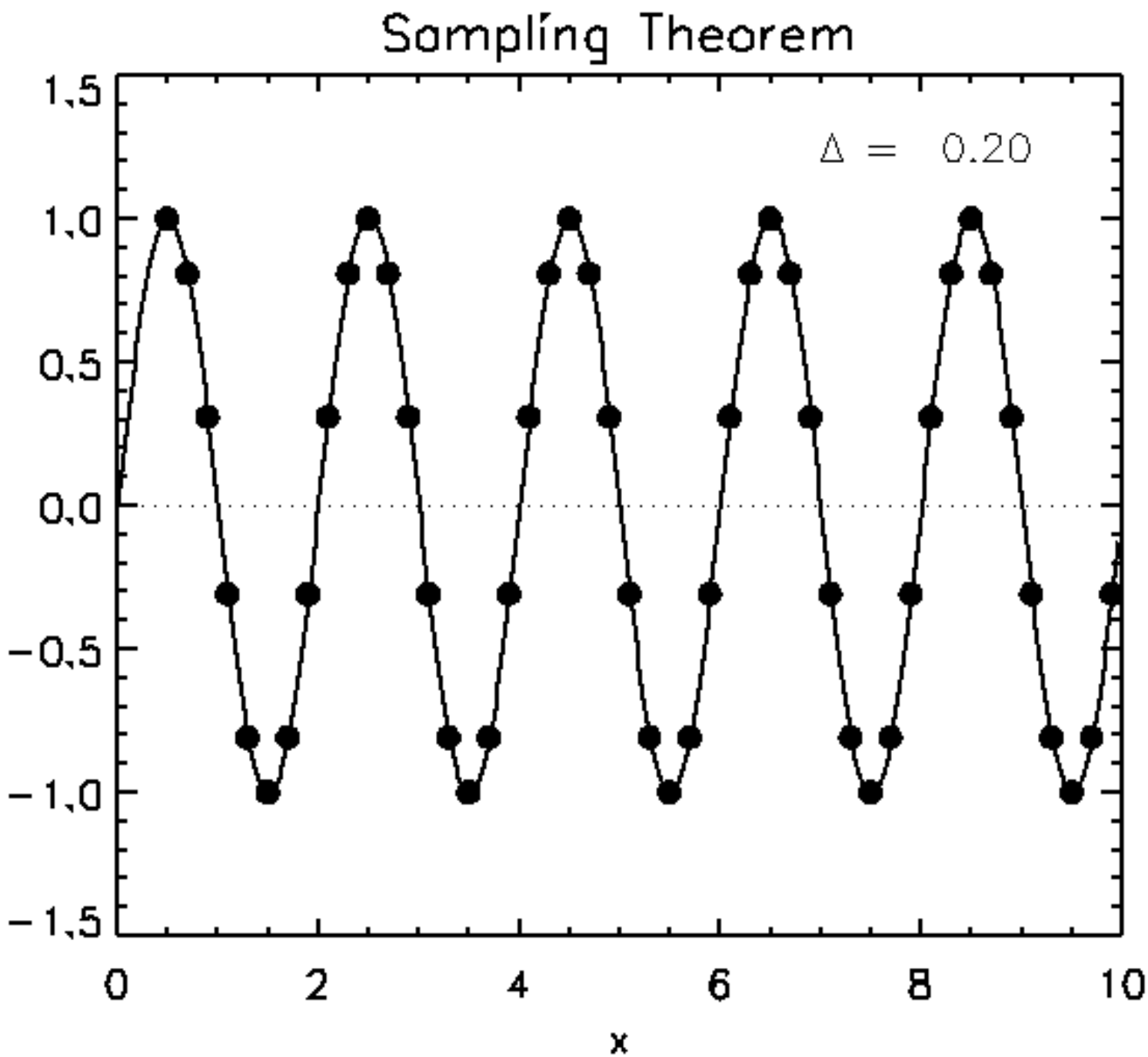
    $$-f_c \leq f \leq f_c$$

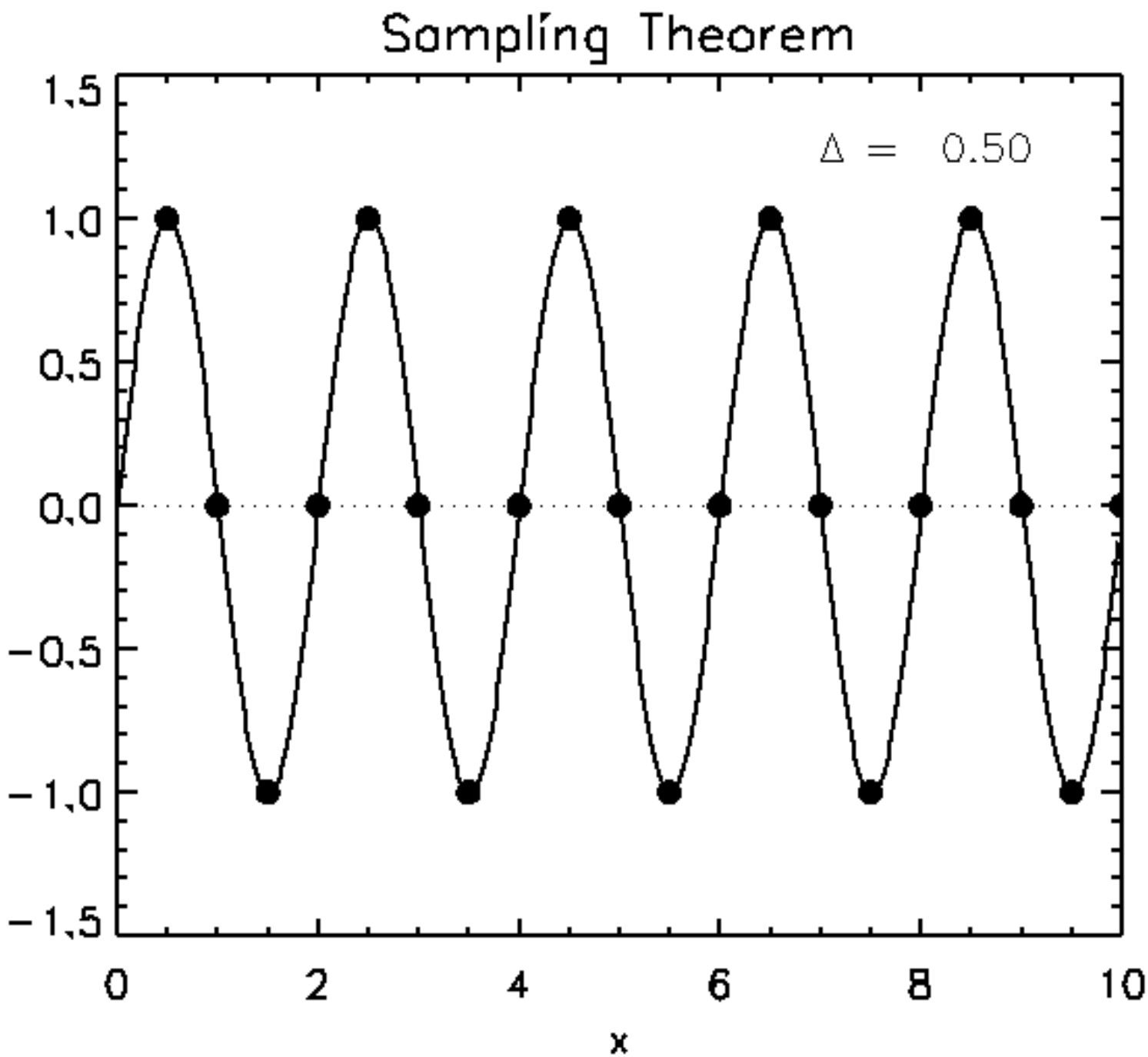
    is spuriously moved into that range
- This phenomenon is called aliasing

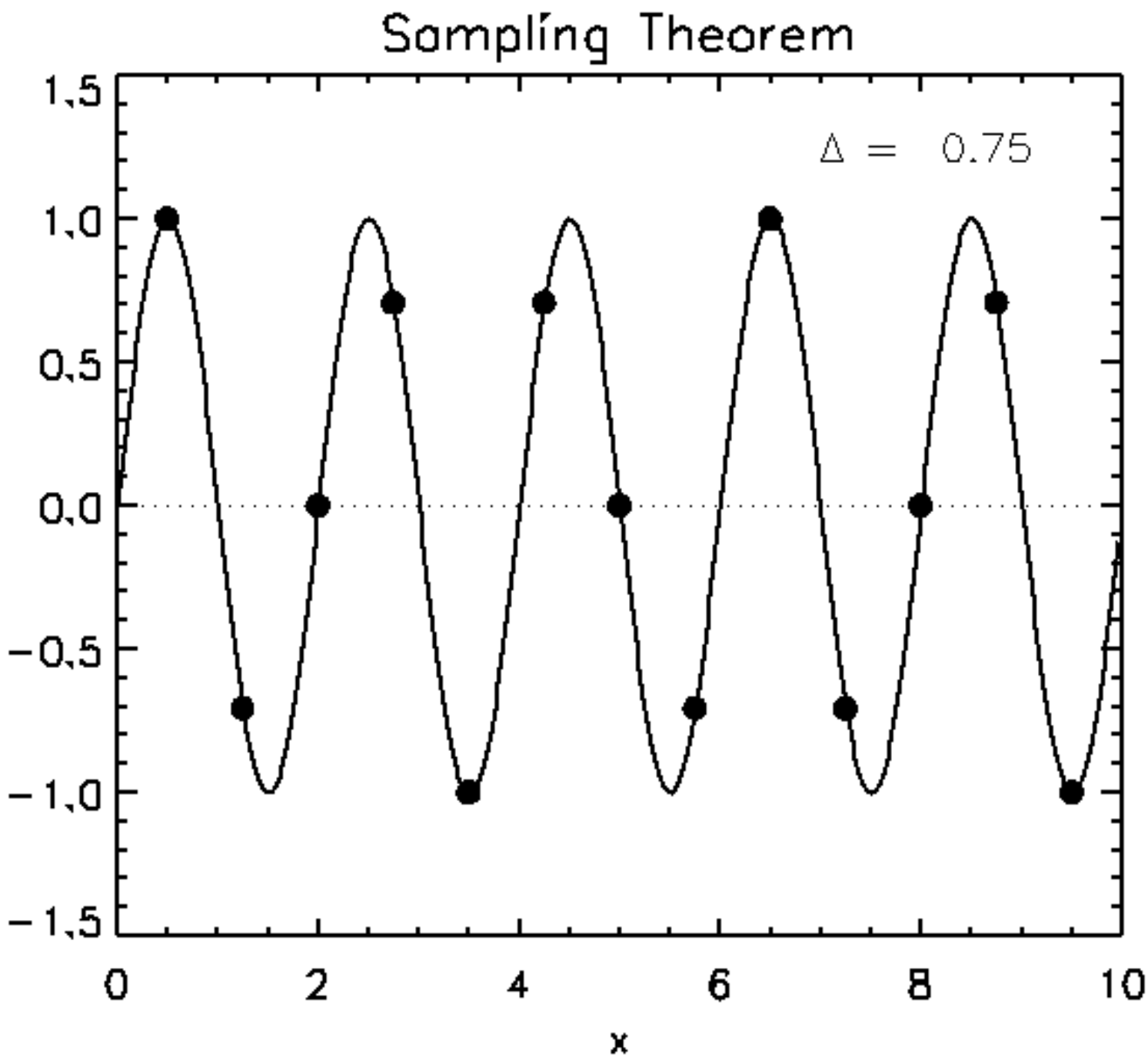


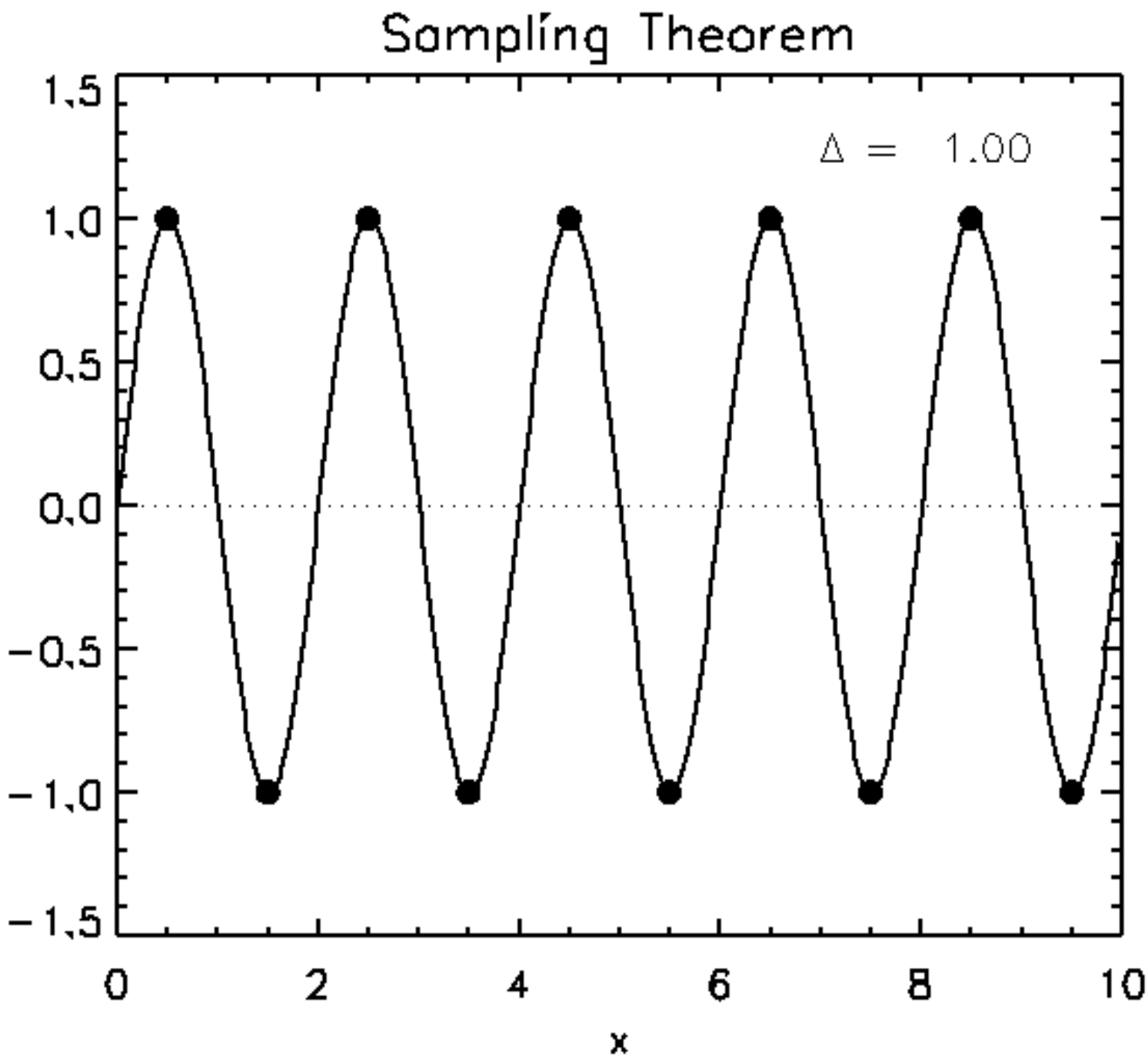
The continuous function, *h(t)*, (a) is nonzero only for a finite interval of time *T*. It follows that its Fourier transform, *H(f)*, whose modulus is shown in (b), is not bandwidth limited but has finite amplitude for all frequencies. If the original function is sampled with a sampling interval Δ, as in (a), then the Fourier transform (c) is defined only between plus and minus the Nyquist critical frequency. Power outside that range is folded over or "aliased" into the range. The effect can be eliminated only by low-pass filtering the original function before sampling.

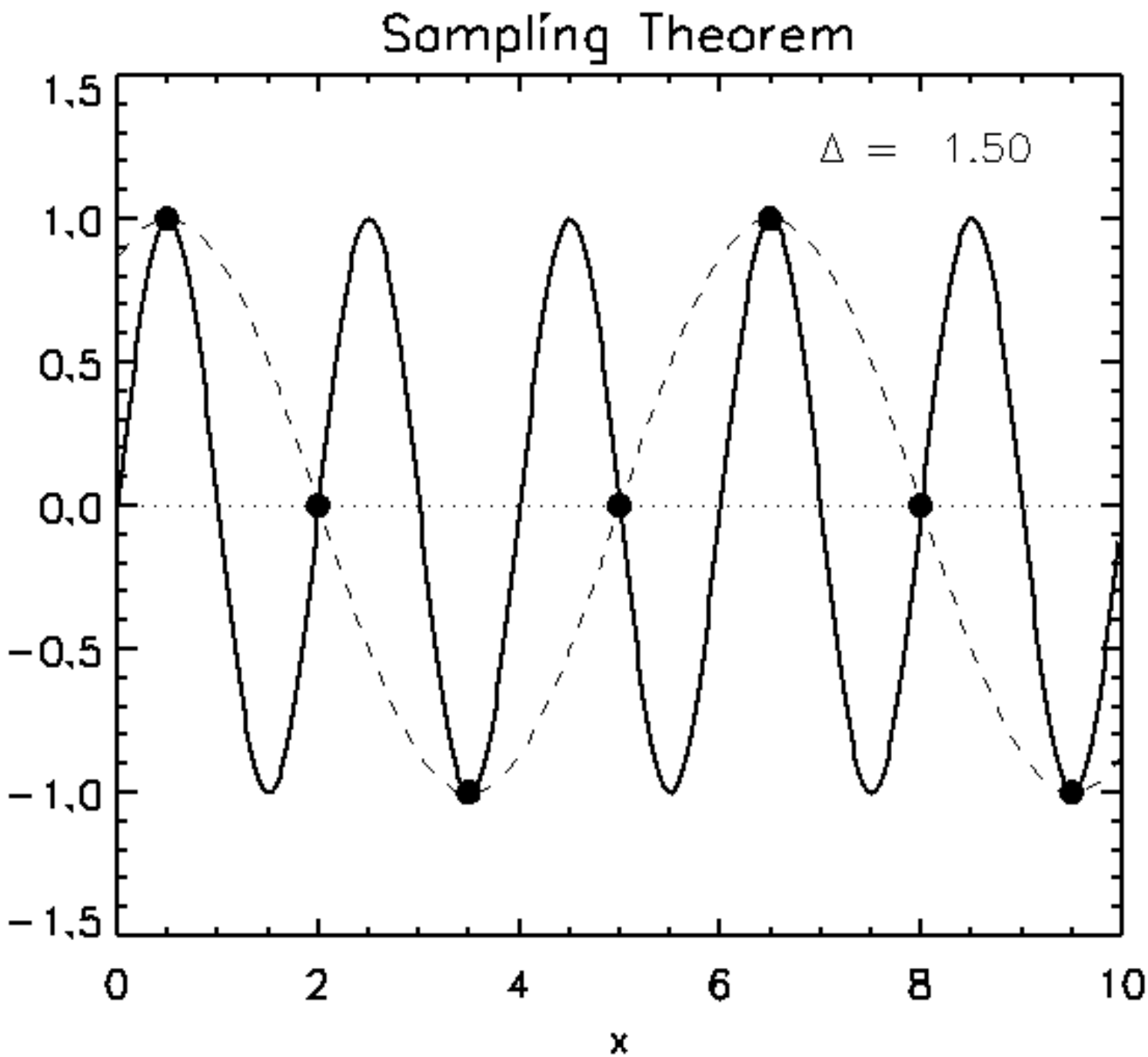

Sampling Theorem
Δ = 0.10
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
2
4
6
8
10
x


Sampling Theorem
Δ = 0.20
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
2
4
6
8
10
x


Sampling Theorem
Δ = 0.50
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
2
4
6
8
10
x


Sampling Theorem
Δ = 0.75
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
2
4
6
8
10
x


Sampling Theorem
Δ = 1.00
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
2
4
6
8
10
x


Sampling Theorem
Δ = 1.50
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
0
2
4
6
8
10
x

Aliasing: The Glass is Half Empty

- There is little that you can do to remove aliased power once you have discretely sampled a signal
- The way to overcome aliasing is to
  - Know the natural bandwidth limit of the signal
  - Enforce a known limit by filtering of the continuous signal, and then sample at a rate sufficiently rapid to give at least two points per cycle of the highest frequency present

Aliasing: The Glass is Half Full

- If a continuous function has been correctly sampled, we can assume that its Fourier transform is equal to zero outside of the frequency range

$$-f_c \leq f \leq f_c$$

- Then we look to at *H(f)* to tell whether the continuous function has been competently sampled (aliasing effects minimized)
- We then look to see whether $H(f) \rightarrow 0$ as $f \rightarrow f_c$
  - If *H(f)* tends to some finite value, then chances are that components outside of the range have been aliased

Optical Systems & Sampling

- An optical systems of lenses and mirrors can be thought of a low pass filter
- The highest spatial frequency present in an image formed by a perfect optical system with a circular pupil is limited by diffraction to

$$f_c = 1/(F\,\lambda)$$

  Where $F = f/D$ is the F number
- e.g., $F/2.5$ Canon lens and $\lambda = 633$ nm

$$f_c = 0.63\ \mu\text{m}^{-1}$$

- Our CCD has $\Delta = 13\ \mu$m, so $1/\Delta = 0.078\ \mu\text{m}^{-1}$