A Little Error Propagation: finding the error in $\langle x \rangle$

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The Centroid

- The centroid of a star is the “center of light”

$$\langle x \rangle = \frac{\sum_i x_i I_i}{\sum_i I_i}$$

but what’s the error in $\langle x \rangle$…

Error propagation

- If $y = f(x_1, x_2, x_3\ldots)$ The fundamental law of error propagation is

$$\sigma_y^2 = \sigma_{x_1}^2\left(\frac{\partial f}{\partial x_1}\right)^2 + \sigma_{x_2}^2\left(\frac{\partial f}{\partial x_2}\right)^2 + \sigma_{x_3}^2\left(\frac{\partial f}{\partial x_3}\right)^2 + \ldots$$

For a quantity where the errors in $x_1, x_2\ldots$ are uncorrelated

- If we apply this to the formula for $\langle x \rangle$

$$\sigma_{\langle x \rangle}^2 = \sigma_{I_1}^2 \left(\frac{\partial \langle x \rangle}{\partial I_1} \right)^2 + \sigma_{I_2}^2 \left(\frac{\partial \langle x \rangle}{\partial I_2} \right)^2 + \sigma_{I_3}^2 \left(\frac{\partial \langle x \rangle}{\partial I_3} \right)^2 + \ldots$$

$$= \sum_i \sigma_{I_i}^2 \left(\frac{\partial \langle x \rangle}{\partial I_i} \right)^2$$

- So the tricky part is computing

$$\frac{\partial\langle x\rangle}{\partial I_j} = \frac{\partial}{\partial I_j}\left(\frac{\sum_i x_i I_i}{\sum_i I_i}\right)$$

Using the chain rule

$$\frac{\partial\langle x\rangle}{\partial I_j} = \frac{\partial}{\partial I_j}\left(\frac{\sum_i x_i I_i}{\sum_i I_i}\right)$$

$$= \sum_i x_i I_i (-1)\left(\sum_i I_i\right)^{-2} \frac{\partial}{\partial I_j}\left(\sum_i I_i\right) + \left(\sum_i I_i\right)^{-1} \frac{\partial}{\partial I_j}\left(\sum_i x_i I_i\right)$$

$$= -\frac{\sum_i x_i I_i}{\left(\sum_i I_i\right)^2}\delta_{ij} + \frac{1}{\sum_i I_i} x_i \delta_{ij}$$

$$= \frac{1}{\sum_i I_i}\left(x_j - \langle x\rangle\right)$$

Putting it all together

$$\sigma_{\langle x\rangle}^2 = \sum_j \sigma_{I_j}^2 \left(\frac{\partial\langle x\rangle}{\partial I_j} \right)^2$$

$$= \sum_j \sigma_{I_j}^2 \left(\frac{1}{\sum_i I_i} \left(x_j - \langle x\rangle \right) \right)^2$$

$$= \left(\sum_i I_i \right)^{-2} \sum_j \sigma_{I_j}^2 \left(x_j - \langle x\rangle \right)^2$$

For Poisson statistics, *i.e.*, if I_i is in photoelectrons

$$\sigma_{\langle x \rangle}^2 = \left(\sum_i I_i \right)^{-2} \sum_j I_j \left(x_j - \langle x \rangle \right)^2$$